66. $x = \sin\theta,\ dx = \cos\theta\, d\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$1 - x^2 = \cos^2\theta$

$$\int \frac{4x^2\,dx}{(1-x^2)^{3/2}} = \int \frac{4\sin^2\theta\cos\theta}{|\cos^3\theta|}\,d\theta$$
$$= \int \frac{4(1-\cos^2\theta)}{\cos^2\theta}\,d\theta$$
$$= \int (4\sec^2\theta - 4)\,d\theta$$
$$= 4\tan\theta - 4\theta + C$$
$$= \frac{4x}{\sqrt{1-x^2}} - 4\sin^{-1}x + C$$

Use Figure 8.18(b) with $a = 1$.

67. For $x \ge 0$, $y \ge 0$ on $(0, 1]$.

$$\text{Volume} = \int_0^1 \pi(-\ln x)^2\,dx$$
$$= \pi\int_0^1 (\ln x)^2\,dx$$
$$= \pi \lim_{b\to 0^+} \int_b^1 (\ln x)^2\,dx$$

Evaluate $\int (\ln x)^2\,dx$ by using integration by parts.

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x}\,dx \qquad v = x$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$$

Evaluate $\int 2\ln x\,dx$ by using integration by parts.

$u = 2\ln x \qquad dv = dx$

$du = \frac{2}{x}\,dx \qquad v = x$

$$\int 2\ln x\,dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x + C$$
$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C$$

$$\text{Area} = \pi \lim_{b\to 0^+}\Big[x(\ln x)^2 - 2x\ln x + 2x\Big]_b^1$$
$$= \pi \lim_{b\to 0^+} [2 - b(\ln b)^2 + 2b\ln b - 2b]$$
$$= 2\pi - \lim_{b\to 0^+}\frac{\pi(\ln b)^2}{1/b} + 2\lim_{b\to 0^+}\frac{\pi\ln b}{1/b}$$
$$= 2\pi - \lim_{b\to 0^+}\frac{2\pi(\ln b)(1/b)}{-1/b^2} + 2\lim_{b\to 0^+}\frac{\pi/b}{-1/b^2}$$
$$= 2\pi - \lim_{b\to 0^+}\frac{2\pi(\ln b)}{-1/b} + 2\lim_{b\to 0^+}(-\pi b)$$
$$= 2\pi - \lim_{b\to 0^+}\frac{2\pi/b}{1/b^2} + 2\pi - \lim_{b\to 0^+} 2\pi b = 2\pi$$

68. For $x \ge 0$, $y \ge 0$ on $[0, \infty)$.

$$\text{Area} = \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$

Evaluate $\int xe^{-x}\,dx$ by using integration by parts.

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$$\text{Area} = \lim_{b\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^b$$
$$= \lim_{b\to\infty}[-be^{-b} - e^{-b} + 1]$$
$$= -\lim_{b\to\infty}\frac{b}{e^b} + 1$$
$$= -\lim_{b\to\infty}\frac{1}{e^b} + 1 = 1$$

69. (a) $\frac{dx}{dt} = k(a-x)^2$

$$\frac{dx}{(a-x)^2} = k\,dt$$
$$\int \frac{dx}{(a-x)^2} = \int k\,dt = kt + C_1$$
$$\frac{1}{a-x} + C_2 = kt + C_1$$
$$\frac{1}{a-x} = kt + C$$

Substitute $x = 0$, $t = 0$

$$\frac{1}{a} = C$$
$$\frac{1}{a-x} = kt + \frac{1}{a}$$
$$\frac{1}{kt + 1/a} = a - x$$
$$x = a - \frac{1}{kt + 1/a}$$

69. continued

(b) $\dfrac{dx}{(a-x)(b-x)} = k\,dt$

$\displaystyle\int \frac{dx}{(a-x)(b-x)} = \int k\,dt = kt + C_1$

$\dfrac{1}{(a-x)(b-x)} = \dfrac{A}{a-x} + \dfrac{B}{b-x}$

$1 = A(b-x) + B(a-x)$

$= (-A-B)x + bA + aB$

Equating coefficients of like terms gives

$-A - B = 0$ and $bA + aB = 1$

Solving the system simultaneously yields

$A = -\dfrac{1}{a-b},\ B = \dfrac{1}{a-b}$

$\displaystyle\int \frac{dx}{(a-x)(b-x)} = \int \frac{-1/(a-b)}{a-x}\,dx + \int \frac{1/(a-b)}{b-x}\,dx$

$= \dfrac{\ln|a-x|}{a-b} - \dfrac{\ln|b-x|}{a-b} + C_2$

$= \dfrac{1}{a-b}\ln\left|\dfrac{a-x}{b-x}\right| + C_2$

$\dfrac{1}{a-b}\ln\left|\dfrac{a-x}{b-x}\right| + C_2 = kt + C_1$

$\ln\left|\dfrac{a-x}{b-x}\right| = (a-b)kt + C$

$\dfrac{a-x}{b-x} = De^{(a-b)kt}$

Substitute $t = 0, x = 0$.

$\dfrac{a}{b} = D$

$\dfrac{a-x}{b-x} = \dfrac{a}{b}e^{(a-b)kt}$

$ab - bx = abe^{(a-b)kt} - axe^{(a-b)kt}$

$x(ae^{(a-b)kt} - b) = ab(e^{(a-b)kt} - 1)$

$x = \dfrac{ab(e^{(a-b)kt} - 1)}{ae^{(a-b)kt} - b}$

Multiply the rational expression by $\dfrac{e^{bkt}}{e^{bkt}}$.

$x = \dfrac{ab(e^{akt} - e^{bkt})}{ae^{akt} - be^{bkt}}$

Chapter 9
Infinite Series

■ Section 9.1 Power Series (pp. 457–468)

Exploration 1 Power Series for Other Functions

1. $1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots$.
2. $x - x^2 + x^3 - x^4 + \cdots + (-1)^n x^{n+1} + \cdots$.
3. $1 + 2x + 4x^2 + 8x^3 + \cdots + (2x)^n + \cdots$.
4. $1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n + \cdots$.
5. $\frac{1}{3} - \frac{1}{3}(x-1) + \frac{1}{3}(x-1)^2 - \frac{1}{3}(x-1)^3 + \cdots + \left(-\frac{1}{3}\right)^n(x-1)^n + \cdots$.

   This geometric series converges for $-1 < x - 1 < 1$, which is equivalent to $0 < x < 2$. The interval of convergence is $(0, 2)$.

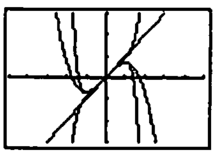
Exploration 2 A Power Series for $\tan^{-1} x$

1. $1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$.
2. $\tan^{-1} x = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt$

   $= \displaystyle\int_0^x (1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots)\,dt$

   $= \left[t - \dfrac{t^3}{3} + \dfrac{t^5}{5} - \dfrac{t^7}{7} + \cdots + (-1)^n\dfrac{t^{2n+1}}{2n+1} + \cdots\right]_0^x$

   $= x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots + (-1)^n\dfrac{x^{2n+1}}{2n+1}\cdots$.
3. The graphs of the first four partial sums appear to be converging on the interval $(-1, 1)$.



[−5, 5] by [−3, 3]

4. When $x = 1$, the series becomes

   $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots + \dfrac{(-1)^n}{2n+1} + \cdots$.

   This series does appear to converge. The terms are getting smaller, and because they alternate in sign they cause the partial sums to oscillate above and below a limit. The two calculator statements shown below will cause the successive partial sums to appear on the calculator each time the ENTER button is pushed. The partial sums will appear to be approaching a limit of $\pi/4$ (which is $\tan^{-1}(1)$), although very slowly.

```
0→N:1→T
                     1
N+1→N:T+(-1)^N/(
2N+1)

█
```

Exploration 3 A Series with a Curious Property

1. $f'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$.

2. $f(0) = 1 + 0 + 0 + \cdots = 1$.

3. Since this function is its own derivative and takes on the value 1 at $x = 0$, we suspect that it must be e^x.

4. If $y = f(x)$, then $\frac{dy}{dx} = y$ and $y = 1$ when $x = 0$.

5. The differential equation is separable.

$$\frac{dy}{y} = dx$$

$$\int \frac{dy}{y} = \int dx$$

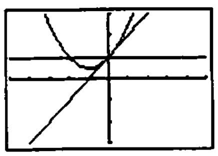
$$\ln |y| = x + C$$

$$y = Ke^x$$

$$1 = Ke^0 \Rightarrow K = 1$$

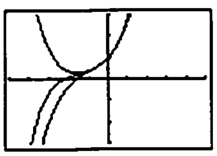
$$\therefore y = e^x.$$

6. The first three partial sums are shown in the graph below. It is risky to draw any conclusions about the interval of convergence from just three partial sums, but so far the convergence to the graph of $y = e^x$ only looks good on $(-1, 1)$. Your answer might differ.



$[-5, 5]$ by $[-3, 3]$

7. The next three partial sums show that the convergence extends outside the interval $(-1, 1)$ in both directions, so $(-1, 1)$ was apparently an underestimate. Your answer in #6 might have been better, but unless you guessed "all real numbers," you still underestimated! (See Example 3 in Section 9.3.)



$[-5, 5]$ by $[-3, 3]$

Quick Review 9.1

1. $u_1 = \frac{4}{1+2} = \frac{4}{3}$

$u_2 = \frac{4}{2+2} = \frac{4}{4} = 1$

$u_3 = \frac{4}{3+2} = \frac{4}{5}$

$u_4 = \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3}$

$u_{30} = \frac{4}{30+2} = \frac{4}{32} = \frac{1}{8}$

2. $u_1 = \frac{(-1)^1}{1} = -1$

$u_2 = \frac{(-1)^2}{2} = \frac{1}{2}$

$u_3 = \frac{(-1)^3}{3} = -\frac{1}{3}$

$u_4 = \frac{(-1)^4}{4} = \frac{1}{4}$

$u_{30} = \frac{(-1)^{30}}{30} = \frac{1}{30}$

3. (a) Since $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$, the common ratio is 3.

(b) $2(3^9) = 39{,}366$

(c) $a_n = 2(3^{n-1})$

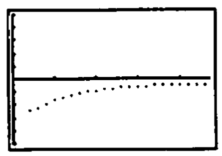
4. (a) Since $\frac{-4}{8} = \frac{2}{-4} = \frac{-1}{2} = -\frac{1}{2}$, the common ratio is $-\frac{1}{2}$.

(b) $8\left(-\frac{1}{2}\right)^9 = -\frac{1}{64}$

(c) $a_n = 8\left(-\frac{1}{2}\right)^{n-1} = 8(-0.5)^{n-1}$

5. (a) We graph the points $\left(n, \frac{1-n}{n^2}\right)$ for $n = 1, 2, 3, \ldots$.

(Note that there is a point at $(1, 0)$ that does not show in the graph.)



$[0, 25]$ by $[-0.5, 0.5]$

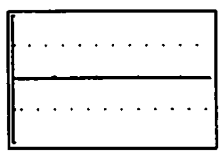
(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1-n}{n^2} = 0$

6. (a) We graph the points $\left(n, \left(1 + \frac{1}{n}\right)^n\right)$ for $n = 1, 2, 3, \ldots$.

$[0, 23.5]$ by $[-1, 4]$

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

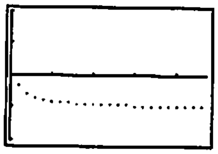
7. (a) We graph the points $(n, (-1)^n)$ for $n = 1, 2, 3, \ldots$.



$[0, 23.5]$ by $[-2, 2]$

(b) $\lim_{n\to\infty} a_n$ does not exist because the values of a_n oscillate between -1 and 1.

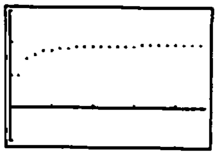
8. (a) We graph the points $\left(n, \frac{1-2n}{1+2n}\right)$ for $n = 1, 2, 3, \ldots$.



[0, 23.5] by [−2, 2]

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1-2n}{1+2n} = -\frac{2}{2} = -1$

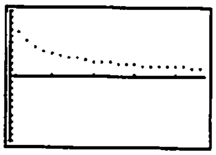
9. (a) We graph the points $\left(n, 2 - \frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$.



[0, 23.5] by [−1, 3]

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(2 - \frac{1}{n}\right) = 2$

10. (a) We graph the points $\left(n, \frac{\ln(n+1)}{n}\right)$ for $n = 1, 2, 3, \ldots$.



[0, 23.5] by [−1, 1]

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n+1)}{n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)}}{1} = 0$

Section 9.1 Exercises

1. (a) Let u_n represent the value of * in the n^{th} term, starting with $n = 1$. Then $\frac{1}{u_1} = 1$, $-\frac{1}{u_2} = -\frac{1}{4}$, $\frac{1}{u_3} = \frac{1}{9}$, and $-\frac{1}{u_4} = -\frac{1}{16}$, so $u_1 = 1$, $u_2 = 4$, $u_3 = 9$, and $u_4 = 16$. We may write $u_n = n^2$, or $* = n^2$.

(b) Let u_n represent the value of * in the n^{th} term, starting with $n = 0$. Then $\frac{1}{u_0} = 1$, $-\frac{1}{u_1} = -\frac{1}{4}$, $\frac{1}{u_2} = \frac{1}{9}$, and $-\frac{1}{u_3} = -\frac{1}{16}$, so $u_0 = 1$, $u_1 = 4$, $u_2 = 9$, and $u_3 = 16$. We may write $u_n = (n+1)^2$, or $* = (n+1)^2$.

(c) If $* = 3$, the series is

$(-1)^3\left(\frac{-1}{1^2}\right) + (-1)^4\left(\frac{-1}{2^2}\right) + (-1)^5\left(\frac{-1}{3^2}\right) + (-1)^6\left(\frac{-1}{4^2}\right) + \cdots$, which is the same as the desired series. Thus let $* = 3$.

2. (a) Note that $a_0 = 1$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{9}$, and so on. Thus $a_n = \left(\frac{1}{3}\right)^n$.

(b) Note that $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, and so on. Thus $a_n = \frac{(-1)^{n-1}}{n}$.

(c) Note that $a_0 = 5$, $a_1 = 0.5$, $a_2 = 0.05$, and so on. Thus $a_n = 5(0.1)^n = \frac{5}{10^n}$.

3. Different, since the terms of $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$ alternate between positive and negative, while the terms of $\sum_{n=1}^{\infty} -\left(\frac{1}{2}\right)^{n-1}$ are all negative.

4. The same, since both series can be represented as $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

5. The same, since both series can be represented as $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

6. Different, since $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ but $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n-1}} = -1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$.

7. Converges; $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3$

8. Diverges, because the terms do not approach zero.

9. Converges; $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)\left(\frac{2}{3}\right)^n = \frac{\frac{5}{4}}{1 - \frac{2}{3}} = 3\left(\frac{5}{4}\right) = \frac{15}{4}$

10. Diverges, because the common ratio is ≥ 1 and the terms do not approach zero.

11. Diverges, because the terms alternate between 1 and −1 and do not approach zero.

12. Converges; $\sum_{n=0}^{\infty} 3(-0.1)^n = \frac{3}{1 - (-0.1)} = \frac{30}{11}$

13. Converges; $\sum_{n=0}^{\infty} \sin^n\left(\frac{\pi}{4} + n\pi\right)$

$$= 1 + \left(-\frac{1}{\sqrt{2}}\right)^1 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1 - \left(-\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{\sqrt{2} + 1}$$

$$= \frac{\sqrt{2}(\sqrt{2} - 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \frac{2 - \sqrt{2}}{2 - 1} = 2 - \sqrt{2}$$

14. Diverges, because the terms do not approach zero.

15. Converges; since $\frac{e}{\pi} \approx 0.865 < 1$, $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n = \frac{1}{1 - \left(\frac{e}{\pi}\right)}$
$= \frac{\pi}{\pi - e}$

16. Converges; $\sum_{n=0}^{\infty} \frac{5^n}{6^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^n = \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)} = 1$

17. Since $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$, the series converges when $|2x| < 1$ and the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Since the sum of the series is $\frac{1}{1-2x}$, the series represents the function $f(x) = \frac{1}{1-2x}$, $-\frac{1}{2} < x < \frac{1}{2}$.

18. Since $\sum_{n=0}^{\infty} (-1)^n(x+1)^n = \sum_{n=0}^{\infty} [-(x+1)]^n$, the series converges when $|-(x+1)| < 1$ and the interval of convergence is $(-2, 0)$. Since the sum of the series is $\frac{1}{1-[-(x+1)]} = \frac{1}{x+2}$, the series represents the function $f(x) = \frac{1}{x+2}$, $-2 < x < 0$.

19. Since $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n = \sum_{n=0}^{\infty} \left(\frac{3-x}{2}\right)^n$, the series converges when $\left|\frac{3-x}{2}\right| < 1$ and the interval of convergence is $(1, 5)$. Since the sum of the series is $\frac{1}{1 - \frac{(3-x)}{2}} = \frac{2}{x-1}$, the series represents the function $f(x) = \frac{2}{x-1}$, $1 < x < 5$.

20. For $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$, the series converges when $\left|\frac{x-1}{2}\right| < 1$ and the interval of convergence is $(-1, 3)$.

Since the sum of the series is $\frac{3}{1 - \frac{(x-1)}{2}} = \frac{6}{3-x}$, the series represents the function $f(x) = \frac{6}{3-x}$, $-1 < x < 3$.

21. Since $\sum_{n=0}^{\infty} \sin^n x = \sum_{n=0}^{\infty} (\sin x)^n$, the series converges when $|\sin x| < 1$. Thus, the series converges for all values of x except odd integer multiples of $\frac{\pi}{2}$, that is, $x \neq (2k+1)\frac{\pi}{2}$ for integers k. Since the sum of the series is $\frac{1}{1-\sin x}$, the series represents the function $f(x) = \frac{1}{1-\sin x}$, $x \neq (2k+1)\frac{\pi}{2}$.

22. Since $\sum_{n=0}^{\infty} \tan^n x = \sum_{n=0}^{\infty} (\tan x)^n$, the series converges when $|\tan x| < 1$. Thus, the series converges for $-\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$, where k is any integer. Since the sum of the series is $\frac{1}{1-\tan x}$, the series represents the function $f(x) = \frac{1}{1-\tan x}$, $-\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$.

23. (a) Since the terms are all positive and do not approach zero, the partial sums tend toward infinity.
(b) The partial sums are alternately 1 and 0.
(c) The partial sums alternate between positive and negative while their magnitude increases toward infinity.

24. Since $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}} = \sum_{n=0}^{\infty} \left(\frac{e^\pi}{\pi^e}\right)^n$, this is a geometric series with common ratio $r = \frac{e^\pi}{\pi^e} \approx 1.03$, which is greater than one.

25. $\sum_{n=0}^{\infty} x^n = 20$

$\frac{1}{1-x} = 20, |x| < 1$

$1 = 20 - 20x$

$20x = 19$

$x = \frac{19}{20}$

26. One possible answer:

For any real number $a \neq 0$,

use $\frac{a}{2} + \frac{a}{4} + \frac{a}{8} + \frac{a}{16} + \frac{a}{32} + \cdots$. To get 0,
use $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} \cdots$.

27. Assuming the series begins at $n = 1$:

(a) $\sum_{n=1}^{\infty} 2r^{n-1} = \frac{2}{1-r} = 5, |r| < 1$

$2 = 5 - 5r$

$5r = 3$

$r = \frac{3}{5}$

Series: $\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{n-1}$

(b) $\sum_{n=1}^{\infty} \frac{13}{2} r^{n-1} = \frac{\frac{13}{2}}{1-r} = 5, |r| < 1$

$\frac{13}{2} = 5 - 5r$

$5r = -\frac{3}{2}$

$r = -\frac{3}{10}$

Series: $\sum_{n=1}^{\infty} \frac{13}{2}\left(-\frac{3}{10}\right)^{n-1}$

28. Let $a = \frac{21}{100}$ and $r = \frac{1}{100}$, giving

$$0.\overline{21} = 0.21 + 0.21(0.01) + 0.21(0.01)^2 + 0.21(0.01)^3 + \cdots = \sum_{n=0}^{\infty} 0.21(0.01)^n = \frac{0.21}{1 - 0.01} = \frac{0.21}{0.99} = \frac{7}{33}$$

29. Let $a = \frac{234}{1000}$ and $r = \frac{1}{1000}$, giving

$$0.\overline{234} = 0.234 + 0.234(0.001) + 0.234(0.001)^2 + 0.234(0.001)^3 + \ldots = \sum_{n=0}^{\infty} 0.234(0.001)^n = \frac{0.234}{1 - 0.001} = \frac{0.234}{0.999} = \frac{26}{111}$$

30. $$0.\overline{7} = 0.7 + 0.7(0.1) + 0.7(0.1)^2 + 0.7(0.1)^3 + \ldots = \sum_{n=0}^{\infty} 0.7(0.1)^n = \frac{0.7}{1 - 0.1} = \frac{0.7}{0.9} = \frac{7}{9}$$

31. $$0.\overline{d} = \frac{d}{10}[1 + 0.1 + 0.1^2 + 0.1^3 + \cdots] = \frac{d}{10}\sum_{n=0}^{\infty} (0.1)^n = \frac{d}{10}\,\frac{1}{1 - 0.1} = \frac{d}{10}\,\frac{1}{0.9} = \frac{d}{9}$$

32. $$0.0\overline{6} = 0.06 + 0.06(0.1) \div 0.06(0.1)^2 + 0.06(0.1)^3 + \cdots = \sum_{n=0}^{\infty} 0.06(.1)^n = \frac{0.06}{1 - 0.1} = \frac{0.06}{0.9} = \frac{1}{15}$$

33. $$1.\overline{414} = 1 + 0.414 + 0.414(0.001) + 0.414(0.001)^2 = 1 + \sum_{n=0}^{\infty} 0.414(0.001)^n = 1 + \frac{0.414}{1 - 0.001} = 1 + \frac{46}{111} = \frac{157}{111}$$

34. $$1.24\overline{123} = 1.24 + 0.00123 + 0.00123(0.001) + 0.00123(0.001)^2 + \cdots = 1.24 + \sum_{n=1}^{\infty} \frac{0.00123}{1 - 0.001} = 1.24 + \frac{0.00123}{0.999} = \frac{124}{100} + \frac{41}{33{,}300} = \frac{41{,}333}{33{,}300}$$

35. $$3.\overline{142857} = 3 + 0.142857(1 + 0.000001 + 0.000001^2 + \cdots) = 3 + 0.142857\sum_{n=0}^{\infty} 0.000001^n = 3 + (0.142857)\left(\frac{1}{1 - 0.000001}\right) = 3 + \frac{0.142857}{0.999999} = 3 + \frac{1}{7} = \frac{22}{7}$$

36. $$\text{Total distance} = 4 + 2[4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots] = 4 + 2\sum_{n=0}^{\infty} 2.4(0.6)^n = 4 + 2\cdot\frac{2.4}{1 - 0.6} = 4 + 2\cdot 6 = 16\text{m}$$

37. $$\text{Total time} = \sqrt{\frac{4}{4.9}} + 2\left[\sqrt{\frac{4(0.6)}{4.9}} + \sqrt{\frac{4(0.6)^2}{4.9}} + \sqrt{\frac{4(0.6)^3}{4.9}} + \ldots\right] = \sqrt{\frac{4}{4.9}} + 2\sqrt{\frac{4(0.6)}{4.9}}[1 + \sqrt{0.6} + (\sqrt{0.6})^2 + \cdots] = \sqrt{\frac{4}{4.9}} + 2\sqrt{\frac{4(0.6)}{4.9}}\cdot\frac{1}{1 - \sqrt{0.6}} \approx 7.113 \text{ sec}$$

38. The area of each square is half of the area of the preceding square, so the total of all the areas is $\sum_{n=0}^{\infty} 4\left(\frac{1}{2}\right)^n$
$= \frac{4}{1-\left(\frac{1}{2}\right)} = 8\text{ m}^2.$

39. Total area $= \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2} \cdot \pi\left(\frac{1}{2^n}\right)^2$
$= \sum_{n=1}^{\infty} \frac{\pi}{2} \cdot \left(\frac{1}{2}\right)^n$
$= \sum_{n=0}^{\infty} \frac{\pi}{4}\left(\frac{1}{2}\right)^n$
$= \frac{\frac{\pi}{4}}{1-\left(\frac{1}{2}\right)}$
$= \frac{\pi}{2}$

40. (a) $S - rS = (a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1})$
$\quad -(ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n)$
$= a - ar^n$

(b) Just factor and divide by $1 - r$:

$S - rS = a - ar^n$

$S(1-r) = a - ar^n$

$S = \frac{a - ar^n}{1-r}$

41. Using the notation $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$, the formula from Exercise 40 is $S_n = \frac{a - ar^n}{1-r}$.
If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ and so $\sum_{n=1}^{\infty} ar^{n-1}$
$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \frac{a}{1-r}$.
If $|r| > 1$ or $r = -1$, then r^n has no finite limit as $n \to \infty$, so the expression $\frac{a - ar^n}{1-r}$ has no finite limit and $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

If $r = 1$, then the nth partial sum is na, which goes to $\pm\infty$.

42. Comparing $\frac{1}{1+3x}$ with $\frac{a}{1-r}$, the leading term is $a = 1$ and the common ratio is $r = -3x$.

Series: $1 - 3x + 9x^2 - \cdots + (-3x)^n + \cdots$

Interval: The series converges when $|-3x| < 1$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

43. Comparing $\frac{x}{1-2x}$ with $\frac{a}{1-r}$, the first term is $a = x$ and the common ratio is $r = 2x$.

Series: $x + 2x^2 + 4x^3 + \cdots + 2^{n-1}x^n + \cdots$

Interval: The series converges when $|2x| < 1$, so the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

44. Comparing $\frac{3}{1-x^3}$ with $\frac{a}{1-r}$, the first term is $a = 3$ and the common ratio is $r = x^3$.

Series: $3 + 3x^3 + 3x^6 + \cdots + 3x^{3n} + \cdots$

Interval: The series converges when $|x^3| < 1$, so the interval of convergence is $(-1, 1)$.

45. Comparing $\frac{1}{1+(x-4)}$ with $\frac{a}{1-r}$, the first term is $a = 1$ and the common ratio is $r = -(x-4)$.

Series: $1 - (x-4) + (x-4)^2 - \cdots + (-1)^n(x-4)^n + \cdots$

Interval: The series converges when $|x-4| < 1$, so the interval of convergence is $(3, 5)$.

46. Comparing $\frac{1}{4}\left(\frac{1}{1+(x-1)}\right)$ with $\frac{a}{1-r}$, the first term is $a = \frac{1}{4}$ and the common ratio is $r = -(x-1) = 1 - x$.

Series:
$\frac{1}{4} - \frac{1}{4}(x-1) + \frac{1}{4}(x-1)^2 - \cdots + \frac{1}{4}(-1)^n(x-1)^n + \cdots$

Interval: The series converges when $|x-1| < 1$, so the interval of convergence is $(0, 2)$.

47. Rewriting $\frac{1}{2-x}$ as $\frac{1}{1-(x-1)}$ and comparing with $\frac{a}{1-r}$, The first term is $a = 1$ and the common ratio is $r = x - 1$.

Series: $1 + (x-1) + (x-1)^2 + \cdots + (x-1)^n + \cdots$

Interval: The series converges when $|x-1| < 1$, so the interval of convergence is $(0, 2)$.

Alternate solution:

Rewriting $\frac{1}{2-x}$ as $\frac{1}{2}\left(\frac{1}{1-\left(\frac{x}{2}\right)}\right)$ and comparing with $\frac{a}{1-r}$, the first is $a = \frac{1}{2}$ and the common ratio is $r = \frac{x}{2}$.

Series: $\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \cdots + \frac{1}{2^{n+1}}x^n + \cdots$

Interval: The series converges when $\left|\frac{x}{2}\right| < 1$, so the interval of convergence is $(-2, 2)$.

48. $1 + e^b + e^{2b} + e^{3b} + \cdots = \sum_{n=0}^{\infty} (e^b)^n$
$= \frac{1}{1-e^b} = 9$
$1 = 9 - 9e^b$
$9e^b = 8$
$e^b = \frac{8}{9}$
$b = \ln\left(\frac{8}{9}\right) = \ln 8 - \ln 9$

49. (a) When $t = 1$, $S = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \left(\frac{1}{2}\right)} = 2.$

(b) S converges when $\left|\frac{t}{1+t}\right| < 1$, or $|t| < |1 + t|$.

For $t < -1$, this inequality is equivalent to

$-t < -(1 + t)$, which is always false.

For $-1 \le t < 0$, the inequality is equivalent to

$-t < 1 + t$, which is true when $t > -\frac{1}{2}$.

For $t \ge 0$, the inequality is equivalent to $t < 1 + t$,

which is always true.

Thus, S converges for all $t > -\frac{1}{2}$.

(c) For $t > -\frac{1}{2}$, we have

$S = \sum_{n=0}^{\infty} \left(\frac{t}{1+t}\right)^n = \frac{1}{1 - \frac{t}{1+t}} = \frac{1+t}{(1+t)-t} = 1 + t$, so

$S > 10$ when $t > 9$.

50. (a) Comparing $f(t) = \frac{4}{1+t^2}$ with $\frac{a}{1-r}$, the first term is $a = 4$ and the common ratio is $r = -t^2$.

First four terms: $4 - 4t^2 + 4t^4 - 4t^6$

General term: $(-1)^n(4t^{2n})$

(b) Note that $G(0) = 0$, so the constant term of the power series for $G(x)$ will be 0. Integrate the terms for $f(x)$ to obtain the terms for $G(x)$.

First four terms: $4x - \frac{4}{3}x^3 + \frac{4}{5}x^5 - \frac{4}{7}x^7$

General term: $(-1)^n\left(\frac{4}{2n+1}\right)x^{2n+1}$

(c) The series in part (a) converges when $|-t^2| < 1$, so the interval of convergence is $(-1, 1)$.

(d) The two numbers are $x = \pm 1$, which result in the convergent series

$G(1) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots + (-1)^n\left(\frac{4}{2n+1}\right) + \cdots$

and

$G(-1)$

$= -4 + \frac{4}{3} - \frac{4}{5} + \frac{4}{7} - \cdots + (-1)^{n-1}\left(\frac{4}{2n+1}\right) + \cdots,$

respectively.

51. Since $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n + \cdots$, we may write $\ln x = \int_1^x \frac{1}{t}\,dt$

$= x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^n(x-1)^n}{n}$

52. To determine our starting point, we note that

$\int f(x)\,dx = \int 2(1-x)^{-3}\,dx = (1-x)^{-2} + C.$

Using the result from Example 4, we have:

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$

$\frac{d}{dx}(1-x)^{-2} = \frac{d}{dx}(1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots)$

$2(1-x)^{-3} = 0 + 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$

Thus, $f(x) = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$.

Replacing n by $n + 2$, this may be written as

$f(x) = 2 + 6x + 12x^2 + (n+2)(n+1)x^n + \cdots$.

Interval: The series converges when $|x| < 1$, so the interval of convergence is $(-1, 1)$.

53. (a) No, because if you differentiate it again, you would have the original series for f, but by Theorem 1, that would have to converge for $-2 < x < 2$, which contradicts the assumption that the original series converges only for $-1 < x < 1$.

(b) No, because if you integrate it again, you would have the original series for f, but by Theorem 2, that would have to converge for $-2 < x < 2$, which contradicts the assumption that the original series converges only for $-1 < x < 1$.

54. Let $L = \lim_{n\to\infty} a_n$. Then by definition of convergence, for $\frac{\epsilon}{2}$ there corresponds an N such that for all m and n,

$n, m > N \Rightarrow |a_m - L| < \frac{\epsilon}{2}$ and $|a_n - L| < \frac{\epsilon}{2}$.

Now,

$|a_m - a_n| = |a_m - L + L - a_n|$

$\le |a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

whenever $m > N$ and $n > N$.

55. Given an $\epsilon > 0$, by definition of convergence there corresponds an N such that for all $n < N$, $|L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$. (There is one such number for each series, and we may let N be the larger of the two numbers.) Now

$$\begin{aligned}|L_2 - L_1| &= |L_2 - a_n + a_n - L_1| \\ &\le |L_2 - a_n| + |a_n - L_1| \\ &< \epsilon + \epsilon \\ &= 2\epsilon.\end{aligned}$$

$|L_2 - L_1| < 2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ. The only nonnegative number smaller than every positive number is 0, so $|L_2 - L_1| = 0$ or $L_1 = L_2$.

56. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $\lim_{n\to\infty} a_{k(n)} = L_1$, $\lim_{n\to\infty} a_{i(n)} = L_2$, and $L_1 \ne L_2$. Given an $\epsilon > 0$ there corresponds an N_1 such that for $k(n) > N_1$, $|a_{k(n)} - L_1| < \epsilon$, and an N_2 such that for $i(n) > N_2$, $|a_{i(n)} - L_2| < \epsilon$. Assume a_n converges.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$, we have that $|a_n - L_1| < \epsilon$ and $|a_n - L_2| < \epsilon$ for infinitely many n. This implies that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$ where $L_1 \ne L_2$. Since the limit of a sequence is unique (by Exercise 55), a_n does not converge and hence diverges.

57. (a) $\lim_{n\to\infty} \dfrac{3n+1}{n+1} = 3$

(b) The line $y = 3$ is a horizontal asymptote of the graph of the function $f(x) = \dfrac{3x+1}{x+1}$, which means $\lim_{x\to\infty} f(x) = 3$. Because $f(n) = a_n$ for all positive integers n, it follows that $\lim_{n\to\infty} a_n$ must also be 3.

■ Section 9.2 Taylor Series (pp. 469–479)

Exploration 1 Designing a Polynomial to Specifications

1. Since $P(0) = 1$, we know that the constant coefficient is 1.

Since $P'(0) = 2$, we know that the coefficient of x is 2.

Since $P''(0) = 3$, we know that the coefficient of x^2 is $\frac{3}{2}$. (The 2 in the denominator is needed to cancel the factor of 2 that results from differentiating x^2.) Similarly, we find the coefficients of x^3 and x^4 to be $\frac{4}{6}$ and $\frac{5}{24}$.

Thus, $P(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{4}{6}x^3 + \frac{5}{24}x^4$.

Exploration 2 A Power Series for the Cosine

1. $\cos(0) = 1$

$\cos'(0) = -\sin(0) = 0$

$\cos''(0) = -\cos(0) = -1$

$\cos^{(3)}(0) = \sin(0) = 0$

etc.

The pattern 1, 0, −1, 0 will repeat forever. Therefore,

$P_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$, and the Taylor series is

$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$.

2. A clever shortcut is simply to differentiate the previously-discovered series for $\sin x$ term-by-term!

Exploration 3 Approximating sin 13

1. 0.4201670368...

4. 20 terms.

Quick Review 9.2

1. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

2. $f(x) = \dfrac{1}{x-1}$

$f'(x) = -(x-1)^{-2}$

$f''(x) = 2(x-1)^{-3}$

$f'''(x) = -6(x-1)^{-4}$

$f^{(n)}(x) = (-1)^n n!(x-1)^{-(n-1)}$

3. $f(x) = 3^x$

$f'(x) = 3^x \ln 3$

$f''(x) = 3^x(\ln 3)^2$

$f'''(x) = 3^x(\ln 3)^3$

$f^{(n)}(x) = 3^x(\ln 3)^n$

4. $f(x) = \ln x$

$f'(x) = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(4)}(x) = -6x^{-4}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ for $n \ge 1$

5. $f(x) = x^n$

$f'(x) = nx^{n-1}$

$f''(x) = n(n-1)x^{n-2}$

$f'''(x) = n(n-1)(n-2)x^{n-3}$

$f^{(k)}(x) = \dfrac{n!}{(n-k)!}x^{n-k}$

$f^{(n)}(x) = \dfrac{n!}{0!}x^0 = n!$

6. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x^n}{n!} = \dfrac{nx^{n-1}}{n!} = \dfrac{x^{n-1}}{(n-1)!}$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{2^n(x-a)^n}{n!} = \dfrac{2^n n(x-a)^{n-1}}{n!} = \dfrac{2^n(x-a)^{n-1}}{(n-1)!}$

8. $\frac{dy}{dx} = \frac{d}{dx}\left[(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right] = (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \frac{(-1)^n x^{2n}}{(2n)!}$

9. $\frac{dy}{dx} = \frac{d}{dx}\frac{(x+a)^{2n}}{(2n)!} = \frac{2n(x+a)^{2n-1}}{(2n)!} = \frac{(x+a)^{2n}}{(2n-1)!}$

10. $\frac{dy}{dx} = \frac{d}{dx}\frac{(1-x)^n}{n!} = \frac{n(1-x)^{n-1}(-1)}{n!} = -\frac{(1-x)^{n-1}}{(n-1)!}$

Section 9.2 Exercises

1. Substitute $2x$ for x in the Maclaurin series for $\sin x$ shown at the end of Section 9.2.

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots + (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} + \cdots$$
$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots + \frac{(-1)^n(2x)^{2n+1}}{(2n+1)!} + \cdots$$

This series converges for all real x.

2. Substitute $-x$ for x in the Maclaurin series for $\ln(1+x)$ shown at the end of Section 9.2.

$$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \cdots + (-1)^{n-1}\frac{(-x)^n}{n} + \cdots$$
$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots$$

This series converges when $-1 \le -x < 1$, so the interval of convergence is $[-1, 1)$.

3. Substitute x^2 for x in the Maclaurin series for $\tan^{-1} x$ shown at the end of Section 9.2.

$$\tan^{-1} x^2 = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \cdots + (-1)^n \frac{(x^2)^{2n+1}}{2n+1} + \cdots$$
$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \cdots + \frac{(-1)^n x^{4n+2}}{2n+1} + \cdots$$

This series converges when $|x^2| \le 1$, so the interval of convergence is $[-1, 1]$.

4. $7xe^x = 7x\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right)$
$$= 7x + 7x^2 + \frac{7x^3}{2!} + \cdots + \frac{7x^{n+1}}{n!}$$

This series converges for all real x.

5. $\cos(x+2) = (\cos 2)(\cos x) - (\sin 2)(\sin x)$

$$= (\cos 2)\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right]$$
$$- (\sin 2)\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right]$$
$$= (\cos 2) - (\sin 2)x - \frac{(\cos 2)x^2}{2!} + \frac{(\sin 2)x^3}{3!} + \frac{(\cos 2)x^4}{4!}$$
$$- \frac{(\sin 2)x^5}{5!} - \cdots$$

We need to write an expression for the coefficient of x^k.

If k is even, the coefficient is $\frac{(-1)^n(\cos 2)}{(2n)!}$ where $2n = k$.

Thus the coefficient is

$\frac{(-1)^{k/2}(\cos 2)}{k!}$, which is the same as $\frac{(-1)^{\text{int}[(k+1)/2]}(\cos 2)}{k!}$.

If k is odd, the coefficient is $\frac{(-1)^{n+1}(\sin 2)}{(2n+1)!}$ where

$2n + 1 = k$. Thus the coefficient is

$\frac{(-1)^{(k+1)/2}(\sin 2)}{(2n+1)!}$, which is the same as $\frac{(-1)^{\text{int}[(k+1)/2]}(\cos 2)}{k!}$.

Hence the general term is $\frac{(-1)^A B x^n}{n!}$, where $A = \text{int}\left(\frac{n+1}{2}\right)$,

and $B = \sin 2$ if n is even and

$B = \cos 2$ if n is odd.

Another way to handle the general term is to observe that

$-\sin 2 = \cos\left(2 + \frac{\pi}{2}\right)$, $-\cos 2 = \cos(2 + \pi)$,

and so on, so the general term is $\left[\frac{1}{n!}\cos\left(2 + \frac{n\pi}{2}\right)\right]x^n$.

The series converges for all real x.

6. $x^2 \cos x = x^2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right)$
$$= x^2 - \frac{x^4}{2} + \frac{x^6}{24} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n)!} + \cdots$$

The series converges for all real x.

7. Factor out x and substitute x^3 for x in the Maclaurin series for $\frac{1}{1-x}$ shown at the end of Section 9.2.

$$\frac{x}{1-x^3} = x\left(\frac{1}{1-x^3}\right)$$
$$= x[1 + x^3 + (x^3)^2 + \cdots + (x^3)^n + \cdots]$$
$$= x + x^4 + x^7 + \cdots + x^{3n+1} + \cdots$$

The series converges for $|x^3| < 1$, so the interval of convergence is $(-1, 1)$.

8. Substitute $-2x$ for x in the Maclaurin series for e^x shown at the end of Section 9.2.

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \cdots + \frac{(-2x)^n}{n!} + \cdots$$
$$= 1 - 2x + 2x^2 - \cdots + \frac{(-1)^n 2^n x^n}{n!} + \cdots$$

The series converges for all real x.

9. $f(2) = \left.\frac{1}{x}\right|_{x=2} = \frac{1}{2}$

$f'(2) = \left.-x^{-2}\right|_{x=2} = -\frac{1}{4}$

$f''(2) = \left.2x^{-3}\right|_{x=2} = \frac{1}{4}$, so $\frac{f''(2)}{2!} = \frac{1}{8}$

$f'''(2) = \left.-6x^{-4}\right|_{x=2} = -\frac{3}{8}$, so $\frac{f'''(2)}{3!} = -\frac{1}{16}$

$P_0(x) = \frac{1}{2}$

$P_1(x) = \frac{1}{2} - \frac{x-2}{4}$

$P_2(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8}$

$P_3(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16}$

10. $f\left(\frac{\pi}{4}\right) = \sin x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right) = \cos x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right) = -\sin x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$, so $\frac{f''\left(\frac{\pi}{4}\right)}{2!} = -\frac{\sqrt{2}}{4}$

$f'''\left(\frac{\pi}{4}\right) = -\cos x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$, so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!} = -\frac{\sqrt{2}}{12}$

$P_0(x) = \frac{\sqrt{2}}{2}$

$P_1(x) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right)$

$P_2(x) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2$

$P_3(x) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2 - \left(\frac{\sqrt{2}}{12}\right)\left(x - \frac{\pi}{4}\right)^3$

11. $f\left(\frac{\pi}{4}\right) = \cos x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right) = -\sin x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right) = -\cos x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$, so $\frac{f''\left(\frac{\pi}{4}\right)}{2!} = -\frac{\sqrt{2}}{4}$

$f'''\left(\frac{\pi}{4}\right) = \sin x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$, so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!} = \frac{\sqrt{2}}{12}$

$P_0(x) = \frac{\sqrt{2}}{2}$

$P_1(x) = \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right)$

$P_2(x) = \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2$

$P_3(x) = \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\sqrt{2}}{12}\right)\left(x - \frac{\pi}{4}\right)^3$

12. $f(4) = x^{1/2}\Big|_{x=4} = 2$

$f'(4) = \frac{1}{2}x^{-1/2}\Big|_{x=4} = \frac{1}{4}$

$f''(4) = -\frac{1}{4}x^{-3/2}\Big|_{x=4} = -\frac{1}{32}$, so $\frac{f''(4)}{2!} = -\frac{1}{64}$

$f'''(4) = \frac{3}{8}x^{-5/2}\Big|_{x=4} = \frac{3}{256}$, so $\frac{f'''(4)}{3!} = \frac{1}{512}$

$P_0(x) = 2$

$P_1(x) = 2 + \frac{x-4}{4}$

$P_2(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64}$

$P_3(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}$

13. (a) Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.
$P_3(x) = x^3 - 2x + 4$ or $4 - 2x + x^3$

(b) $f(1) = x^3 - 2x + 4\Big|_{x=1} = 3$

$f'(1) = 3x^2 - 2\Big|_{x=1} = 1$

$f''(1) = 6x\Big|_{x=1} = 6$, so $\frac{f''(1)}{2!} = 3$

$f'''(1) = 6\Big|_{x=1} = 6$, so $\frac{f'''(1)}{3!} = 1$

$P_3(x) = 3 + (x-1) + 3(x-1)^2 + (x-1)^3$

14. (a) Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.
$P_3(x) = 2x^3 + x^2 + 3x - 8$ or $-8 + 3x + x^2 + 2x^3$

(b) $f(1) = 2x^3 + x^2 + 3x - 8\Big|_{x=1} = -2$

$f'(1) = 6x^2 + 2x + 3\Big|_{x=1} = 11$

$f''(1) = 12x + 2\Big|_{x=1} = 14$, so $\frac{f''(1)}{2!} = 7$

$f'''(1) = 12\Big|_{x=1} = 12$, so $\frac{f'(1)}{3!} = 2$

$P_3(x) = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$

15. (a) Since $f(0) = f'(0) = f''(0) = f'''(0) = 0$, the Taylor polynomial of order 3 is $P_3(0) = 0$.

(b) $f(1) = x^4\Big|_{x=1} = 1$

$f'(1) = 4x^3\Big|_{x=1} = 4$

$f''(1) = 12x^2\Big|_{x=1} = 12$, so $\frac{f''(1)}{2!} = 6$

$f'''(1) = 24x\Big|_{x=1} = 24$, so $\frac{f'''(1)}{3!} = 4$

$P_3(x) = 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3$

16. (a) $P_3(x) = 4 + 5x + \frac{-8}{2!}x^2 + \frac{6}{3!}x^3$
$= 4 + 5x - 4x^2 + x^3$

$f(0.2) \approx P_3(0.2) = 4.848$

16. continued

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $5 - 8x + 3x^2$. Evaluating at $x = 0.2$,

$f'(0.2) \approx 3.52$

17. (a) $P_3(x) = 4 + (-1)(x-1) + \frac{3}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$

$= 4 - (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

$f(1.2) \approx P_3(1.2) \approx 3.863$

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $-1 + 3(x-1) + (x-1)^2$. Evaluating at $x = 1.2$,

$f'(1.2) \approx -0.36$

18. (a) Since $f'(0)x = \frac{x}{2!}$, $f'(0) = \frac{1}{2!} = \frac{1}{2}$.

Since $\frac{f^{(10)}(0)}{10!}x^{10} = \frac{x^{10}}{11!}$, $f^{(10)}(0) = \frac{10!}{11!} = \frac{1}{11}$.

(b) Multiply each term of $f(x)$ by x.

$g(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$

(c) $g(x) = e^x - 1$

19. (a) Substitute $\frac{x}{2}$ for x in the Maclaurin series for e^x shown at the end of Section 9.2

$$e^{x/2} = 1 + \frac{x}{2} + \frac{\left(\frac{x}{2}\right)^2}{2} + \cdots + \frac{\left(\frac{x}{2}\right)^n}{n!} + \cdots$$

$$= 1 + \frac{x}{2} + \frac{x^2}{8} + \cdots + \frac{x^n}{2^n \cdot n!}$$

(b) $g(x) = \frac{e^x - 1}{x}$

$$= \frac{1}{x}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right) - 1\right]$$

$$= \frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$

$$= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^{n-1}}{n!} + \cdots$$

This can also be written as

$$1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^n}{(n+1)!} + \cdots.$$

(c) $g'(x) = \frac{d}{dx}\frac{e^x - 1}{x} = \frac{(x)(e^x) - (e^x - 1)(1)}{x^2}$

$= \frac{xe^x - e^x + 1}{x^2}$

$g'(1) = \frac{e - e + 1}{1} = 1$

From the series,

$$g'(x) = \frac{d}{dx}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots\right)$$

$$= \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \cdots + \frac{nx^{n-1}}{(n+1)!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!}$$

Therefore, $g'(1) = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$, which means $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

20. (a) Factor out 2 and substitute t^2 for x in the Maclaurin series for $\frac{1}{1-x}$ at the end of Section 9.2.

$f(t) = \frac{2}{1-t^2}$

$= 2\left(\frac{1}{1-t^2}\right)$

$= 2[1 + t^2 + (t^2)^2 + (t^2)^3 + \cdots + (t^2)^n + \cdots]$

$= 2 + 2t^2 + 2t^4 + 2t^6 + \cdots + 2t^{2n} + \cdots$

(b) Since $G(0) = 0$, the constant term is zero and we may find $G(x)$ by integrating the terms of the series for $f(x)$.

$$G(x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots$$

21. (a) $f(0) = (1+x)^{1/2}\Big|_{x=0} = 1$

$f'(0) = \frac{1}{2}(1+x)^{-1/2}\Big|_{x=0} = \frac{1}{2}$

$f''(0) = -\frac{1}{4}(1+x)^{-3/2}\Big|_{x=0} = -\frac{1}{4}$, so $\frac{f''(0)}{2!} = -\frac{1}{8}$

$f'''(0) = \frac{3}{8}(1+x)^{-5/2}\Big|_{x=0} = \frac{3}{8}$, so $\frac{f'''(0)}{3!} = \frac{1}{16}$

$P_4(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$

(b) Since $g(x) = f(x^2)$, the first four terms are $1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16}$.

(c) Since $h(0) = 5$, the constant term is 5. The next three terms are obtained by integrating the first three terms of the answer to part (b). The first four terms of the series for $h(x)$ are $5 + x + \frac{x^3}{6} - \frac{x^5}{40}$.

22. **(a)** $a_0 = 1$

$a_1 = \frac{3}{1}a_0 = 3 \cdot 1 = 3$

$a_2 = \frac{3}{2}a_1 = \frac{3}{2} \cdot 3 = \frac{9}{2}$

$a_3 = \frac{3}{3}a_2 = a_2 = \frac{9}{2}$

Since each term is obtained by multiplying the previous term by $\frac{3}{n}$, $a_n = \frac{3^n}{n!}$.

$\sum_{n=0}^{\infty} a_n x^n = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots + \frac{3^n}{n!}x^n + \cdots$

(b) Since the series can be written as $\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$, it represents the function $f(x) = e^{3x}$.

(c) $f'(1) = 3e^{3x}\Big|_{x=1} = 3e^3$

23. First, note that $\cos 18 \approx 0.6603$.

Using $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, enter the following two-step commands on your home screen and continue to hit ENTER.

```
0→N: 1→T
                 1
N+1→N: T+(-1)^N*1
8^(2N)/(2N)!→T
              -161
              4213
          -43026.2
```

The sum corresponding to $N = 25$ is about 0.6582 (not withing 0.001 of exact value), and the sum corresponding to $N = 26$ is about 0.6606, which is within 0.001 of the exact value. Since we began with $N = 0$, it takes a total of 27 terms (or, up to and including the 52nd degree term).

24. One possible answer: Because the end behavior of a polynomial must be unbounded and $\sin x$ is not unbounded. Another: Because $\sin x$ has an infinite number of local extrema, but a polynomial can only have a finite number.

25. (1) $\sin x$ is odd and $\cos x$ is even
(2) $\sin 0 = 0$ and $\cos 0 = 1$

26. Replace x by $3x$ in series for $\sin x$. Therefore, we have $\frac{(3x)^5}{5!}$ so $\frac{3^5}{5!} = \frac{81}{40}$.

27. Since $\frac{d^3}{dx^3} \ln x = 2x^{-3}$, which is $\frac{1}{4}$ at $x = 2$, the coefficient is $\frac{\frac{1}{4}}{3!} = \frac{1}{24}$.

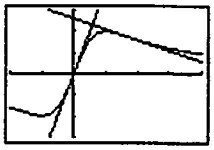
28. The linearization of f at a is the first order Taylor polynomial generated by f at $x = a$.

29. **(a)** Since $f'(x) = \frac{d}{dx}\frac{4x}{x^2+1}$

$= \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2}$

$= \frac{4 - 4x^2}{(x^2+1)^2}$,

we have $f(0) = 0, f'(0) = 4, f(\sqrt{3}) = \sqrt{3}$ and $f'(\sqrt{3}) = -\frac{1}{2}$, so the linearizations are $L_1(x) = 4x$ and $L_2(x) = \sqrt{3} - \frac{1}{2}(x - \sqrt{3}) = -\frac{1}{2}x + \frac{3}{2}\sqrt{3}$, respectively.



[−2, 4] by [−3, 3]

(b) $f''(a)$ must be 0 because of the inflection point, so the second degree term in the Taylor series of f at $x = a$ is zero.

30. The series represents $\tan^{-1} x$. When $x = 1$, it converges to $\tan^{-1} 1 = \frac{\pi}{4}$. When $x = -1$, it converges to $\tan^{-1}(-1) = -\frac{\pi}{4}$.

31. **(a)** $f(x) = \frac{1}{x}(\sin x)$

$= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$

$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$

(b) Because f is undefined at $x = 0$.

(c) $k = 1$

32. Note that the Maclaurin series for $\frac{1}{1-x}$ is $1 + x + x^2 + \cdots + x^n + \cdots$. If we differentiate this series and multiply by x, we obtain the desired Maclaurin series $x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$. Therefore, the desired function is

$f(x) = x\frac{d}{dx}\frac{1}{1-x} = x\frac{1}{(1-x)^2} = \frac{x}{(x-1)^2}$.

33. **(a)** $f(x) = (1+x)^m$

$f'(x) = m(1+x)^{m-1}$

$f''(x) = m(m-1)(1+x)^{m-2}$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$

(b) Differentiating $f(x)$ k times gives
$f^{(k)}(x) = m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}$.
Substituting 0 for x, we have
$f^{(k)}(0) = m(m-1)(m-2)\cdots(m-k+1)$.

33. continued

(c) The coefficient is

$$\frac{f^{(k)}(0)}{k!} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

(d) $f(0) = 1, f'(0) = m$, and we're done by part (c).

34. Because $f(x) = (1 + x)^m$ is a polynomial of degree m. Alternately, observe that $f^{(k)}(0) = 0$ for $k \geq m + 1$.

Section 9.3 Taylor's Theorem (pp. 480–487)

Exploration 1 Your Turn

1. We need to consider what happens to $R_n(x)$ as $n \to \infty$.

By Taylor's Theorem, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - 0)^n$, where $f^{(n+1)}(c)$ is the $(n + 1)$st derivative of $\cos x$ evaluated at some c between x and 0. As with $\sin x$, we can say that $f^{(n+1)}(c)$ lies between -1 and 1 inclusive. Therefore, no matter what x is, we have

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n\right| \leq \left|\frac{1}{(n+1)!}x^n\right| = \frac{|x|^n}{(n+1)!}.$$

The factorial growth in the denominator, as noted in Example 3, eventually outstrips the power growth in the numerator, and we have $\frac{|x|^n}{(n+1)!} \to 0$ for all x. This means that $R_n(x) \to 0$ for all x, which completes the proof.

Exploration 2 Euler's Formula

1.
$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots + \frac{(ix)^n}{n!} + \cdots$$
$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots + (i)^n\frac{x^n}{n!} + \cdots$$

2. If we isolate the terms in the series that have i as a factor, we get:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots + (i)^n\frac{x^n}{n!} + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$$
$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$
$$= \cos x + i \sin x.$$

(We are assuming here that we can rearrange the terms of a convergent series without affecting the sum. It happens to be true in this case, but we will see in Section 9.5 that it is not always true.)

3. $e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0 = -1$
Thus, $e^{i\pi} + 1 = 0$

Quick Review 9.3

1. Since $|f(x)| = |2 \cos (3x)| \leq 2$ on $[-2\pi, 2\pi]$ and $f(0) = 2$, $M = 2$.

2. Since $f(x)$ is increasing and positive on $[1, 2]$, $M = f(2) = 7$.

3. Since $f(x)$ is increasing and positive on $[-3, 0]$, $M = f(0) = 1$.

4. Since the minimum value of $f(x)$ is $f(-1) = -\frac{1}{2}$ and the maximum value of $f(x)$ is $f(1) = \frac{1}{2}$, $M = \frac{1}{2}$.

5. On $[-3, 1]$, the minimum value of $f(x)$ is $f(-3) = -7$ and the maximum value of $f(x)$ is $f(0) = 2$. On $(1, 3]$, f is increasing and positive, so the maximum value of f is $f(3) = 5$. Thus $|f(x)| \leq 7$ on $[-3, 3]$ and $M = 7$.

6. Yes, since each expression for an nth derivative given by the Quotient Rule will be a rational function whose denominator is a power of $x + 1$.

7. No, since the function $f(x) = |x^2 - 4|$ has a corner at $x = 2$.

8. Yes, since the derivatives of all orders for $\sin x$ and $\cos x$ are defined for all values of x.

9. Yes, since the function $f(x) = e^{-x}$ has derivatives of the form $f^{(n)}(x) = -e^{-x}$ for odd values of n and $f^{(n)}(x) = e^{-x}$ for even values of n, and both of these expressions are defined for all values of x.

10. No, since $f(x) = x^{3/2}$, we have $f'(x) = \frac{3}{2}x^{1/2}$ and $f''(x) = \frac{3}{4}x^{-1/2}$, so $f''(0)$ is undefined.

Section 9.3 Exercises

1. $f(0) = e^{-2x}\big|_{x=0} = 1$

$f'(0) = -2e^{-2x}\big|_{x=0} = -2$

$f''(0) = 4e^{-2x}\big|_{x=0} = 4$, so $\frac{f''(0)}{2!} = 2$

$f'''(0) = -8e^{-2x}\big|_{x=0} = -8$, so $\frac{f'''(0)}{3!} = -\frac{4}{3}$

$f^{(4)}(0) = 16e^{-2x}\big|_{x=0} = 16$, so $\frac{f^{(4)}(0)}{4!} = \frac{2}{3}$

$P_4(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4$

$f(0.2) \approx P_4(0.2) = 0.6704$

2. $f(0) = \cos \frac{\pi x}{2}\big|_{x=0} = 1$

$f'(0) = -\frac{\pi}{2} \sin \frac{\pi x}{2}\big|_{x=0} = 0$

$f''(0) = -\frac{\pi^2}{4} \cos \frac{\pi x}{2}\big|_{x=0} = -\frac{\pi^2}{4}$, so $\frac{f''(0)}{2!} = -\frac{\pi^2}{8}$

$f'''(0) = \frac{\pi^3}{8} \sin \frac{\pi x}{2}\big|_{x=0} = 0$, so $\frac{f'''(0)}{3!} = 0$

$f^{(4)}(0) = \frac{\pi^4}{16} \cos \frac{\pi x}{2}\big|_{x=0} = \frac{\pi^4}{16}$, so $\frac{f^{(4)}(0)}{4!} = \frac{\pi^4}{384}$

$P_4(x) = 1 - \frac{\pi^2}{8}x^2 + \frac{\pi^4}{384}x^4$

$f(0.2) \approx P_4(0.2) \approx 0.9511$

3. $f(0) = 5\sin(-x)\Big|_{x=0} = -5\sin x\Big|_{x=0} = 0$

$f'(0) = -5\cos x\Big|_{x=0} = -5$

$f''(0) = 5\sin x\Big|_{x=0} = 0$, so $\frac{f''(0)}{2!} = 0$

$f'''(0) = 5\cos x\Big|_{x=0} = 5$, so $\frac{f'''(0)}{3!} = \frac{5}{6}$

$f^{(4)}(0) = -5\sin x\Big|_{x=0} = 0$, so $\frac{f^{(4)}(0)}{4!} = 0$

$P_4(x) = -5x + \frac{5}{6}x^3$

$f(0.2) \approx P_4(0.2) = -\frac{149}{150} \approx -0.9933$

4. Substituting x^2 for x in the Maclaurin series given for $\ln(1 + x)$ at the end of Section 9.2, we have

$$\ln(1 + x^2) = x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \cdots + (-1)^{n-1}\frac{(x^2)^n}{n} + \cdots$$
$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots + (-1)^{n-1}\frac{x^{2n}}{n} + \cdots$$

Therefore, $P_4(x) = x^2 - \frac{x^4}{2}$ and $f(0.2) \approx P(0.2) = 0.0392$.

5. $f(0) = (1 - x)^{-2}\Big|_{x=0} = 1$

$f'(0) = 2(1 - x)^{-3}\Big|_{x=0} = 2$

$f''(0) = 6(1 - x)^{-4}\Big|_{x=0} = 6$, so $\frac{f''(0)}{2!} = 3$

$f'''(0) = 24(1 - x)^{-5}\Big|_{x=0} = 24$, so $\frac{f'''(0)}{3!} = 4$

$f^{(4)}(0) = 120(1 - x)^{-6}\Big|_{x=0} = 120$, so $\frac{f^{(4)}(0)}{4!} = 5$

$P_4(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$

$f(0.2) \approx P_4(0.2) = 1.56$

6. $$xe^x = x\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$
$$= x + x^2 + \frac{x^3}{2!} + \cdots + \frac{x^{n+1}}{n!} + \cdots$$

7. $$\sin x - x + \frac{x^3}{3!}$$
$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots\right) - x + \frac{x^3}{3!}$$
$$= \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Note: By replacing n with $n + 2$, the general term can be written as $(-1)^n\frac{x^{2n+5}}{(2n+5)!}$

8. $$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$$
$$= \frac{1}{2} + \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n\frac{(2x)^{2n}}{(2n)!} + \cdots\right)$$
$$= 1 - \frac{4x^2}{2 \cdot 2!} + \frac{16x^4}{2 \cdot 4!} - \cdots + (-1)^n\frac{2^{2n}x^{2n}}{2 \cdot (2n)!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{3} - \cdots + (-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots$$

9. $$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$
$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots + (-1)^n\frac{(2x)^{2n}}{(2n)!} + \cdots\right)$$
$$= \frac{4x^2}{2 \cdot 2!} - \frac{16x^4}{2 \cdot 4!} + \frac{64x^6}{2 \cdot 6!} - \cdots + (-1)^{n-1}\frac{2^{2n}x^{2n}}{2 \cdot (2n)!} + \cdots$$
$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots + (-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots$$

Note: By replacing n with $n + 1$, the general term can be written as $(-1)^n\frac{2^{2n+1}x^{2n+2}}{(2n+2)!}$.

10. $$\frac{x^2}{1 - 2x} = x^2\left(\frac{1}{1 - 2x}\right)$$
$$= x^2[1 + 2x + (2x)^2 + \cdots + (2x)^n + \cdots]$$
$$= x^2 + 2x^3 + 4x^4 + \cdots + 2^n x^{n+2} + \cdots$$

11. Let $f(x) = \sin x$. Then $P_4(x) = P_3(x) = x - \frac{x^3}{6}$, so we use the Remainder Estimation Theorem with $n = 4$. Since $|f^{(5)}(x)| = |\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_4(x)| \le \frac{|x|^5}{5!}$, so we may assure that $|R_4(x)| \le 5 \times 10^{-4}$ by requiring $\frac{|x|^5}{5!} \le 5 \times 10^{-4}$, or $|x| \le \sqrt[5]{0.06} \approx 0.5697$. Thus, the absolute error is no greater than 5×10^{-4} when

$-0.56 < x < 0.56$ (approximately).

Alternate method: Using graphing techniques, $\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \le 5 \times 10^{-4}$ when $-0.57 < x < 0.57$.

12. Let $f(x) = \cos x$. Then $P_3(x) = P_2(x) = 1 - \frac{x^2}{2}$, so we may use the Remainder Estimation Theorem with $n = 3$. Since $|f^{(4)}(x)| = |\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_3(x)| \le \frac{|x|^4}{4!}$. For $|x| < 0.5$, the absolute error is less than $\frac{(0.5)^4}{4!} \approx 0.0026$ (approximately).

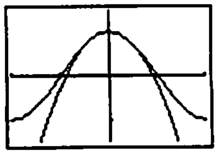
12. continued

Alternate method: Using graphing techniques, we find that

when $|x| < 0.5$,

$$|\text{error}| = \left|\cos x - \left(1 - \frac{x^2}{2}\right)\right|$$
$$< \left|\cos 0.5 - \left(1 - \frac{0.5^2}{2}\right)\right|$$
$$\approx 0.002583.$$

The quantity $1 - \frac{x^2}{2}$ tends to be too small, as shown by the graphs of $y = \cos x$ and $y = 1 - \frac{x^2}{2}$.



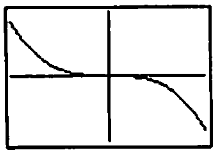
$[-\pi, \pi]$ by $[-1.5, 1.5]$

13. Let $f(x) = \sin x$. Then $P_2(x) = P_1(x) = x$, so we may use the Remainder Estimation Theorem with $n = 2$. Since $|f'''(x)| = |-\cos x| \le 1$ for all x, we may use $M = r = 1$, giving $|R_2(x)| \le \frac{|x|^3}{3!}$. Thus, for $|x| < 10^{-3}$, the maximum possible error is about $\frac{(10^{-3})^3}{3!} \approx 1.67 \times 10^{-10}$.

Alternate method:

Using graphing techniques, we find that when $|x| < 10^{-3}$, $|\text{error}| = |\sin x - x| \le |\sin 10^{-3} - 10^{-3}| \approx 1.67 \times 10^{-10}$.

The inequality $x < \sin x$ is true for $x < 0$, as we may see by graphing $y = \sin x - x$.



$[-10^{-3}, 10^{-3}]$ by $[-2 \times 10^{-10}, 2 \times 10^{-10}]$

14. Let $f(x) = \sqrt{1 + x}$. Then $P_1(x) = 1 + \frac{x}{2}$, so we may use the Remainder Estimation Theorem with $n = 1$. Since $|f''(x)| = \left|-\frac{1}{4}(1 + x)^{-3/2}\right|$, which is less than 0.2538 for $|x| < 0.01$, we may use $M = 0.2538$ and $r = 1$, giving $|R_1(x)| \le \frac{0.2538|x|^2}{2!}$. Thus, for $|x| < 0.01$ the maximum possible absolute error is about

$$\frac{0.2538(0.01)^2}{2!} \approx 1.27 \times 10^{-5}.$$

Alternate method:

Using graphing techniques, we find that when $|x| < 0.01$,

$$|\text{error}| = \left|\sqrt{1 + x} - \left(1 + \frac{x}{2}\right)\right|$$
$$\le \left|\sqrt{1 - 0.01} - \left(1 - \frac{0.01}{2}\right)\right|$$
$$\approx 1.26 \times 10^{-5}.$$

15. Note that $1 + x + \frac{x^2}{2}$ is the second order Taylor polynomial for $f(x) = e^x$ at $x = 0$, so we may use the Remainder Estimation Theorem with $n = 2$. Since $|f'''(x)| = e^x$, which is less than $e^{0.1}$ when $|x| < 0.1$ and $r = 1$, giving $|R_2(x)| \le \frac{e^{0.1}|x|^3}{3!}$. Thus, for $|x| < 0.1$, the maximum possible error is about $\frac{e^{0.1}(0.1)^3}{3!} \approx 1.842 \times 10^{-4}$.

16. Note that $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ and
$e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots + (-1)^n\frac{x^n}{n!} + \cdots$

Thus the terms with n even will cancel for $\sinh x = \frac{1}{2}(e^x - e^{-x})$, and the terms with n odd will cancel for $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n + 1)!} + \cdots$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

17. All of the derivatives of $\cosh x$ are either $\cosh x$ or $\sinh x$. For any real x, $\cosh x$ and $\sinh x$ are both bounded by $e^{|x|}$. So for any real x, let $M = e^{|x|}$ and $r = 1$ in the Remainder Estimation Theorem. This gives $|R_n(x)| \le \frac{e^{|x|}|x|^{n+1}}{(n + 1)!}$. But for any fixed value of x, $\lim_{n\to\infty} \frac{e^{|x|}|x|^{n+1}}{(n + 1)!} = 0$. It follows that the series converges to $\cosh x$ for all real values of x.

18. For $n = 0$, Taylor's Theorem with Remainder says that if f has derivatives of all orders in an open interval I containing a, then for each x in I,
$f(x) = f(a) + R(x)$, where $R(x) = f'(c)(x - a)$, so
$f(x) = f(a) + f'(c)(x - a)$ for some c between a and x.
Letting $b = x$ this equation is $f(b) = f(a) + f'(c)(b - a)$, which is equivalent to $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some c between a and b. Thus, for the class of functions that have derivatives of all orders in an open interval containing a and b, the Mean Value Theorem can be considered a special case of Taylor's Theorem.

19. $f(0) = \ln(\cos x)\Big|_{x=0} = \ln 1 = 0$

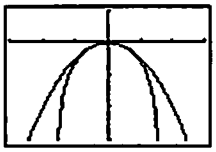
$f'(0) = \frac{1}{\cos x}(-\sin x)\Big|_{x=0} = -\tan x\Big|_{x=0} = 0$

$f''(0) = -\sec^2 x\Big|_{x=0} = -1$ so $\frac{f''(0)}{2!} = -\frac{1}{2}$

(a) $L(x) = 0$

(b) $P_2(x) = -\frac{1}{2}x^2$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−3, 1]

20. $f(0) = e^{\sin x}\Big|_{x=0} = e^0 = 1$

$f'(0) = e^{\sin x}\cos x\Big|_{x=0} = 1$

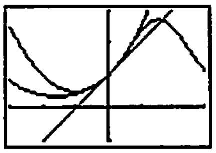
$f''(0) = \left[(e^{\sin x})(-\sin x) + (\cos x)(e^{\sin x}\cos x)\right]\Big|_{x=0} = 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1 + x$

(b) $P_2(x) = 1 + x + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−1, 3]

21. $f(0) = (1 - x^2)^{-1/2}\Big|_{x=0} = 1$

$f'(0) = -\frac{1}{2}(1 - x^2)^{-3/2}(-2x)\Big|_{x=0} = x(1 - x^2)^{-3/2}\Big|_{x=0} = 0$

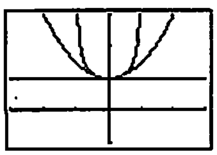
$f''(0) = (x)\left[-\frac{3}{2}(1 - x^2)^{-5/2}(-2x)\right] + (1 - x^2)^{-3/2}\Big|_{x=0} = 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−1, 3]

22. $f(0) = \sec x\Big|_{x=0} = 1$

$f'(0) = \sec x \tan x\Big|_{x=0} = 0$

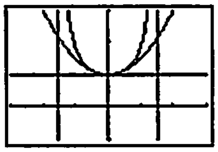
$f''(0) = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)\Big|_{x=0} = 1,$

so $\frac{f''(0)}{2!} = \frac{1}{2}$

(a) $L(x) = 1$

(b) $P_2(x) = 1 + \frac{x^2}{2}$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−1, 3]

23. $f(0) = \tan x\Big|_{x=0} = 0$

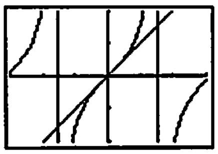
$f'(0) = \sec^2 x\Big|_{x=0} = 1$

$f''(0) = (2\sec x)(\sec x \tan x)\Big|_{x=0} = 0$, so $\frac{f''(0)}{2!} = 0$

(a) $L(x) = x$

(b) $P_2(x) = x$

(c) The graphs of the linear and quadratic approximations fit the graph of the function near $x = 0$.



[−3, 3] by [−2, 2]

24. $f(0) = (1 + x)^k\Big|_{x=0} = 1$

$f'(0) = k(1 + x)^{k-1}\Big|_{x=0} = k$

$f''(0) = k(k - 1)(1 + x)^{k-2}\Big|_{x=0} = k(k - 1),$

so $\frac{f''(0)}{2!} = \frac{k(k - 1)}{2}$

$P_2(x) = 1 + kx + \frac{k(k - 1)}{2}x^2$

For $k = 3$, we have $f(x) = (1 + x)^3$ and $f'''(x) = 6$. We may use the Remainder Estimation Theorem with $n = 2$, $M = 6$, and $r = 1$, giving $R_2(x) \le \frac{6|x|^3}{3!} = |x|^3$. (In this particular case it is actually true that $R_2(x) = x^3$, since $f(x)$ is a cubic polynomial.) Thus the absolute error is less than $\frac{1}{100}$ whenever $|x|^3 < 0.01$. In the interval [0, 1], this occurs when $0 \le x < \sqrt[3]{0.01} \approx 0.215$.

Alternate method:

Note that $P_2(x) = 1 + 3x + 3x^2$. Using graphing techniques,

$|(1 + x)^3 - (1 + 3x + 3x^2)| < \frac{1}{100}$ when $|x| < 0.215$.

25. Let $f(x) = e^x$. Then $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, so we may use the Remainder Estimation Theorem with $n = 3$. Since $|f^{(4)}(x)| = e^x$, which is no more than $e^{0.1}$ when $|x| \le 0.1$, we may use $M = e^{0.1}$ and $r = 1$, giving $|R_3(x)| \le \frac{e^{0.1}|x|^4}{4!}$. Thus, for $|x| \le 0.1$, the maximum possible absolute error is about $\frac{e^{0.1}(0.1)^4}{24} \approx 4.605 \times 10^{-6}$.

Alternate method:

Using graphing techniques, when $|x| \le 0.1$,

$$|\text{error}| = \left|e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\right|$$
$$\le \left|e^{0.1} - \left(1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6}\right)\right|$$
$$\approx 4.251 \times 10^{-6}.$$

26. Since the Maclaurin series is

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots,$$

$P_3(x) = 1 + x + x^2 + x^3$.

Since $|f^{(4)}(x)| = 24(1 - x)^{-5}$, which is no more than $24(0.9)^{-5}$ when $|x| \le 0.1$, we may use $M = 24(0.9)^{-5}$ and $r = 1$, giving $|R_3(x)| \le \frac{24(0.9)^{-5}|x|^4}{4!} = \frac{|x|^4}{0.9^5}$. Thus, for $|x| \le 0.1$, an upper bound for the magnitude of the approximation error is $\frac{0.1^4}{0.9^5} \approx 1.694 \times 10^{-4}$. Rounding up to be safe, an upper bound is 1.70×10^{-4}.

Alternate method:

Using graphing techniques, when $|x| \le 0.1$,

$$|\text{error}| = \left|\frac{1}{1-x} - (1 + x + x^2 + x^3)\right|$$
$$\le \left|\frac{1}{1-0.1} - 1.111\right| \approx 1.11 \times 10^{-4}.$$

27. (a) No

(b) Yes, since

$$\frac{dy}{dx} = e^{-x^2}$$
$$= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{2!} - \cdots + (-1)^n\frac{x^{2n}}{n!} + \cdots.$$

The constant term of y is $y(0) = 2$, and we may obtain the remaining terms of y by integrating the above series.

$$y = 2 + x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

By substituting $n - 1$ for n, the general term may also be written as $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)(n-1)!}$.

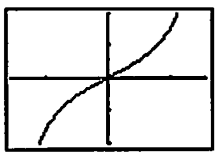
(c) The power series equals the function y for all real values of x. This is because the series for e^{-x^2} converges for all real values of x, so Theorem 2 of Section 9.1 implies that the new series also converges for all x.

28. (a) Substitute $-x$ for x in the Maclaurin series for $\ln(1 + x)$ given at the end of Section 9.2.

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots$$

(b)
$$\ln\frac{1+x}{1-x} = \ln(1 + x) - \ln(1 - x)$$
$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots\right)$$
$$+ \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}\right)$$
$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots$$

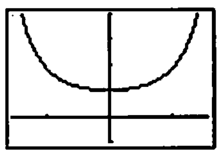
29. (a)



$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-2, 2]$

The series approximates $\tan x$.

(b)



$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $[-1, 4]$

The series approximates $\sec x$.

30. **(a)** $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots + (-1)^n\frac{(2x)^{2n}}{(2n)!} + \cdots\right)$$

$$= \frac{4x^2}{2\cdot 2!} - \frac{16x^4}{2\cdot 4!} + \frac{64x^6}{2\cdot 6!} - \frac{256x^8}{2\cdot 8!} + \frac{1024x^{10}}{2\cdot 10!} - \cdots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \frac{2x^{10}}{14{,}175} - \cdots$$

(b) derivative $= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \cdots$

(c) part (b) $= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots = \sin 2x$

31. **(a)** It works. For example, let $n = 2$. Then $P = 3.14$ and $P + \sin P \approx 3.141592653$, which is accurate to more than 6 decimal places.

(b) Let $P = \pi + x$ where x is the error in the original estimate. Then

$P + \sin P = (\pi + x) + \sin(\pi + x) = \pi + x - \sin x$

But by the Remainder Theorem, $|x - \sin x| < \frac{|x|^3}{6}$.

Therefore, the difference between the new estimate $P + \sin P$ and π is less than $\frac{|x|^3}{6}$.

32. **(a)** $\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{(\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta))}{2}$

$$= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2}$$

$$= \frac{2\cos\theta}{2} = \cos\theta$$

(b) $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{(\cos\theta + i\sin\theta) - (\cos(-\theta) + i\sin(-\theta))}{2i}$

$$= \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{2i}$$

$$= \frac{2i\sin\theta}{2i} = \sin\theta$$

33. $\frac{d}{dx}[e^{ax}(\cos bx + i\sin bx)]$

$= (e^{ax})(-b\sin bx + bi\cos bx) + (ae^{ax})(\cos bx + i\sin bx)$

$= (e^{ax})[(bi^2\sin bx + bi\cos bx) + a(\cos bx + i\sin bx)]$

$= (e^{ax})[bi(\cos bx + i\sin bx) + a(\cos bx + i\sin bx)]$

$= (a + bi)(e^{ax})(\cos bx + i\sin bx)$

$= (a + bi)e^{(a+bi)x}$

34. **(a)** The derivative of the right-hand side is

$$\frac{a - bi}{a^2 + b^2}(a + bi)e^{(a+bi)x}$$

$$= \frac{a^2 - (bi)^2}{a^2 + b^2}e^{(a+bi)x}$$

$$= \frac{a^2 + b^2}{a^2 + b^2}e^{(a+bi)x} = e^{(a+bi)x},$$

which confirms the antiderivative formula.

(b) $\int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx$

$$= \int e^{(a+bi)x}\,dx$$

$$= \frac{a - bi}{a^2 + b^2}e^{(a+bi)x}$$

$$= \frac{a - bi}{a^2 + b^2}e^{ax}(\cos bx + i\sin bx)$$

$$= \left(\frac{e^{ax}}{a^2 + b^2}\right)(a\cos bx + b\sin bx - bi\cos bx + ai\sin bx)$$

$$= \left(\frac{e^{ax}}{a^2 + b^2}\right)[(a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)]$$

Separating the real and imaginary parts gives

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$ and

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$

■ Section 9.4 Radius of Convergence (pp. 487–496)

Exploration 1 Finishing the Proof of the Ratio Test

1. For $\sum\frac{1}{n}$: $L = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1$.

   For $\sum\frac{1}{n^2}$: $L = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$.

2. (a) $\int_1^\infty \frac{1}{x}\,dx = \lim_{k\to\infty}\left(\ln x\Big|_1^k\right) = \lim_{k\to\infty}\ln k = \infty$.

   (b) $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{k\to\infty}\left(-x^{-1}\Big|_1^k\right) = \lim_{k\to\infty}\left(-\frac{1}{k} + 1\right) = 1$.

3. Figure 9.14a shows that $\sum\frac{1}{n}$ is greater than $\int_1^\infty\frac{1}{x}\,dx$. Since the integral diverges, so must the series.

   Figure 9.14b shows that $\sum\frac{1}{n^2}$ is less than $1 + \int_1^\infty\frac{1}{x^2}\,dx$. Since the integral converges, so must the series.

4. These two examples prove that $L = 1$ can be true for either a divergent series or a convergent series. The Ratio Test itself is therefore inconclusive when $L = 1$.

Exploration 2 Revisiting a Maclaurin Series

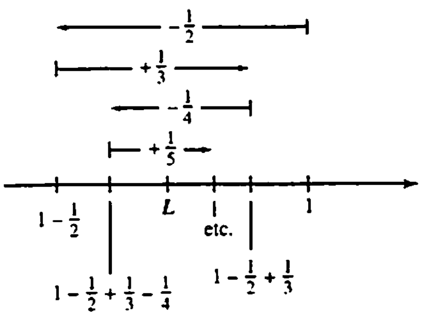
1. $L = \lim_{n\to\infty}\frac{|x|^{n+1}}{n+1}\cdot\frac{n}{|x|^n} = \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|$. The series converges absolutely when $|x| < 1$, so the radius of convergence is 1.

2. When $x = -1$, the series becomes

   $-1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} - \cdots.$

   Each term in this series is the negative of the corresponding term in the divergent series of Figure 9.14a. Just as $\sum\frac{1}{n}$ diverges to $+\infty$, this series diverges to $-\infty$.

3. Geometrically, we chart the progress of the partial sums as in the figure below:



4. The series converges at the right-hand endpoint. As shown in the picture above, the partial sums are closing in on some limit L as they oscillate left and right by constantly decreasing amounts.

5. We know that the series does not converge absolutely at the right-hand endpoint, because $\sum \frac{1}{n}$ diverges (Exploration 1 of this section).

Quick Review 9.4

1. $\lim_{n\to\infty} \frac{n|x|}{n+1} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|$

2. $\lim_{n\to\infty} \frac{n^2|x-3|}{n(n+1)} = |x-3| \lim_{n\to\infty} \frac{n^2}{n^2+n} = |x-3|$

3. $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$
(Note: This limit is similar to the limit which is discussed at the end of Example 3 in Section 9.3.)

4. $\lim_{n\to\infty} \frac{(n+1)^4x^2}{(2n)^4} = x^2 \lim_{n\to\infty} \frac{n^4+4n^3+6n^2+4n+1}{16n^4}$
$= x^2\left(\frac{1}{16}\right) = \frac{x^2}{16}$

5. $\lim_{n\to\infty} \frac{|2x+1|^{n+1}2^n}{2^{n+1}|2x+1|^n} = \lim_{n\to\infty} \frac{|2x+1|}{2} = \frac{|2x+1|}{2}$

6. Since $n^2 > 5n$ for $n \ge 6$, $a_n = n^2$, $b_n = 5n$, and $N = 6$.

7. Since $5^n > n^5$ for $n \ge 6$, $a_n = 5^n$, $b_n = n^5$ and $N = 6$.

8. Since $\sqrt{n} > \ln n$ for $n \ge 1$, $a_n = \sqrt{n}$, $b_n = \ln n$, and $N = 1$.

9. Since $10^n < n!$ $\left(\text{and hence } \frac{1}{10^n} > \frac{1}{n!}\right)$ for $n \ge 25$, $a_n = \frac{1}{10^n}$, $b_n = \frac{1}{n!}$, and $N = 25$.

10. Since $n^2 < n^3$ $\left(\text{and hence } \frac{1}{n^2} > n^{-3}\right)$ for $n \ge 2$, $a_n = \frac{1}{n^2}$, $b_n = n^{-3}$, and $N = 2$.

Section 9.4 Exercises

1. Diverges by the nth-Term Test, since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \ne 0$.

2. Diverges by the nth-Term Test, since $\lim_{n\to\infty} \frac{2^n}{n+1} = \infty$. (The Ratio Test can also be used.)

3. Converges by the Ratio Test, since
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2-1}{2^{n+1}} \cdot \frac{2^n}{n^2-1} = \frac{1}{2} < 1$.

4. Converges, because it is a geometric series with $r = \frac{1}{8}$, so $|r| < 1$.

5. Converges by the Ratio Test, since
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(3^{n+1}+1)} \cdot \frac{3^n+1}{2^n} = \frac{2}{3} < 1$.
Alternately, note that $\frac{2^n}{3^n+1} < \left(\frac{2}{3}\right)^n$ for all n.
Since $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges, $\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$ converges by the Direct Comparison Test.

6. Diverges by the nth-Term Test, since
$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1 \ne 0$

7. Converges by the Ratio Test, since
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2e^{-n-1}}{n^2e^{-n}} = e^{-1} < 1$.

8. Converges by the Ratio Test, since
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}} = \frac{1}{10} < 1$.

9. Converges by the Ratio Test, since
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+4)!}{3!(n+1)!3^{n+1}} \cdot \frac{3!n!\,3^n}{(n+3)!}$
$= \lim_{n\to\infty} \frac{n+4}{3(n+1)}$
$= \frac{1}{3} < 1$.

10. Diverges by the nth-Term Test, since
$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \ne 0$.

11. Converges, because it is a geometric series with $r = -\frac{2}{3}$, so $|r| < 1$.

12. Diverges by the Ratio Test, since
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!e^{-n-1}}{n!e^{-n}} = \lim_{n\to\infty} (n+1)e^{-1} = \infty$.
(The nth-Term Test can also be used.)

13. Diverges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}}{(n+1)^3\,2^{n+1}}\cdot\frac{n^3 2^n}{3^n}$$
$$=\lim_{n\to\infty}\frac{3n^3}{(n+1)^3(2)}$$
$$=\frac{3}{2}>1.$$

(The nth = Term Test can also be used.)

14. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{2^{n+1}}\cdot\frac{2^n}{n\ln n}$$
$$=\lim_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{\ln(n+1)}{\ln n}$$
$$=\frac{1}{2}<1.$$

15. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!}{(2n+3)!}\cdot\frac{(2n+1)!}{n!}$$
$$=\lim_{n\to\infty}\frac{n+1}{(2n+3)(2n+2)}$$
$$=\lim_{n\to\infty}\frac{1}{2(2n+3)}=0<1.$$

16. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}$$
$$=\lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)(n+1)^n}$$
$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$
$$=\lim_{n\to\infty}\frac{1}{(1+1/n)^n}$$
$$=\frac{1}{e}<1$$

17. One possible answer:

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (see Exploration 1 in this section) even though $\lim_{n\to\infty}\frac{1}{n}=0$.

18. One possible answer:

Let $a_n=2^{-n}$ and $b_n=3^{-n}$

Then $\sum a_n$ and $\sum b_n$ are convergent geometric series, but $\sum\frac{a_n}{b_n}=\sum\left(\frac{3}{2}\right)^n$ is a divergent geometric series.

19. This is a geometric series which converges only for $|x|<1$, so the radius of convergence is 1.

20. This is a geometric series which converges only for $|x+5|<1$, so the radius of convergence is 1.

21. This is a geometric series which converges only for $|-(4x+1)|<1$, or $\left|x+\frac{1}{4}\right|<\frac{1}{4}$, so the radius of convergence is $\frac{1}{4}$.

22. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|3x-2|^{n+1}}{n+1}\cdot\frac{n}{|3x-2|^n}=|3x-2|$

The series converges for $|3x-2|<1$, or $\left|x-\frac{2}{3}\right|<\frac{1}{3}$, and diverges for $\left|x-\frac{2}{3}\right|>\frac{1}{3}$, so the radius of convergence is $\frac{1}{3}$.

23. This is a geometric series which converges only for $\left|\frac{x-2}{10}\right|<1$, or $|x-2|<10$, so the radius of convergence is 10.

24. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{n+3}\cdot\frac{n+2}{n|x|^n}=\lim_{n\to\infty}|x|=|x|$

The series converges for $|x|<1$ and diverges for $|x|>1$, so the radius of convergence is 1.

25. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}}\cdot\frac{n\sqrt{n}\,3^n}{|x|^n}$

$=\lim_{n\to\infty}\frac{|x|}{3}=\frac{|x|}{3}$

The series converges for $|x|<3$ and diverges for $|x|>3$, so the radius of convergence is 3.

26. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|x|^{2n+3}}{(n+1)!}\cdot\frac{n!}{|x|^{2n+1}}=\lim_{n\to\infty}\frac{x^2}{n+1}=0$

The series converges for all values of x, so the radius of convergence is ∞.

27. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+1)|x+3|^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n|x+3|^n}$

$=\lim_{n\to\infty}\frac{|x+3|}{5}=\frac{|x+3|}{5}$

The series converges for $|x+3|<5$ and diverges for $|x+3|>5$, so the radius of convergence is 5.

28. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{4^{n+1}[(n+1)^2+1]}\cdot\frac{4^n(n^2+1)}{n|x|^n}$

$=\lim_{n\to\infty}\frac{|x|}{4}=\frac{|x|}{4}$

The series converges for $|x|<4$ and diverges for $|x|>4$, so the radius of convergence is 4.

29. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{\sqrt{n+1}\,|x|^{n+1}}{3^{n+1}}\cdot\frac{3^n}{\sqrt{n}\,|x|^n}=\lim_{n\to\infty}\frac{|x|}{3}=\frac{|x|}{3}$

The series converges for $|x|<3$ and diverges for $|x|>3$, so the radius of convergence is 3.

30. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x-4|^{n+1}}{n!|x-4|^n}$

$= \lim_{n\to\infty}(n+1)|x-4|$

$= \infty\ (x \neq 4)$

The series converges only for $x = 4$, so the radius of convergence is 0.

31. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|(-2)^{n+1}|(n+2)|x-1|^{n+1}}{|-2^n|(n+1)|x-1|^n}$

$= \lim_{n\to\infty} 2|x-1|$

$= 2|x-1|$

The series converges for $|x-1| < \frac{1}{2}$ and diverges for $|x-1| > \frac{1}{2}$, so the radius of convergence is $\frac{1}{2}$.

32. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|4x-5|^{2n+3}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{|4x-5|^{2n+1}}$

$= \lim_{n\to\infty}(4x-5)^2$

$= (4x-5)^2$

The series converges for $(4x-5)^2 < 1$, which is equivalent to $|4x-5| < 1$, or $\left|x-\frac{5}{4}\right| < \frac{1}{4}$ and diverges for $\left|x-\frac{5}{4}\right| > \frac{1}{4}$. The radius of convergence is $\frac{1}{4}$.

33. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x+\pi|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x+\pi|^n}$

$= \lim_{n\to\infty}|x+\pi|$

$= |x+\pi|$

The series converges for $|x+\pi| < 1$ and diverges for $|x+\pi| > 1$, so the radius of convergence is 1.

34. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-\sqrt{2}|^{2n+3}}{2^{n+1}}\cdot\frac{2^n}{|x-\sqrt{2}|^{2n+1}}$

$= \lim_{n\to\infty}\frac{1}{2}(x-\sqrt{2})^2$

$= \frac{1}{2}(x-\sqrt{2})^2$

The series converges for $\frac{1}{2}(x-\sqrt{2})^2 < 1$, which is equivalent to $|x-\sqrt{2}| < \sqrt{2}$, and diverges for $|x-\sqrt{2}| > \sqrt{2}$. The radius of convergence is $\sqrt{2}$.

35. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x-1)^2}{4}$. It converges only when $\left|\frac{(x-1)^2}{4}\right| < 1$, so the interval of convergence is $-1 < x < 3$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{4}{4-(x-1)^2} = \frac{4}{-x^2+2x+3} = -\frac{4}{x^2-2x-3}$$

36. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x+1)^2}{9}$. It converges only when $\left|\frac{(x+1)^2}{9}\right| < 1$, so the interval of convergence is $-4 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x+1)^2}{9}} = \frac{9}{9-(x+1)^2} = \frac{9}{-x^2-2x+8} = -\frac{9}{x^2+2x-8}$$

37. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{\sqrt{x}}{2} - 1$. It converges only when $\left|\frac{\sqrt{x}}{2} - 1\right| < 1$, so the interval of convergence is $0 < x < 16$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\left(\frac{\sqrt{x}}{2}-1\right)} = \frac{2}{4-\sqrt{x}}$$

38. This is a geometric series with first term $a = 1$ and common ratio $r = \ln x$. It converges only when $|\ln x| < 1$, so the interval of convergence is $\frac{1}{e} < x < e$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\ln x}$$

39. This is a geometric series with first term $a = 1$ and common ratio $\frac{x^2-1}{3}$. It converges only when $\left|\frac{x^2-1}{3}\right| < 1$, so the interval of convergence is $-2 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{x^2-1}{3}} = \frac{3}{3-(x^2-1)} = \frac{3}{4-x^2}$$

40. This is a geometric series with first term $a = 1$ and common ratio $\frac{\sin x}{2}$. Since $\left|\frac{\sin x}{2}\right| < 1$ for all x, the interval of convergence is $-\infty < x < \infty$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{\sin x}{2}} = \frac{2}{2-\sin x}$$

41. Almost, but the Ratio Test won't determine whether there is convergence or divergence at the endpoints of the interval.

42. **(a)** For $k \le N$, it's obvious that

$$a_1 + \cdots + a_k \le a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n.$$

For all $k > N$,

$$a_1 + \cdots + a_k = a_1 + \cdots + a_N + a_{N+1} + \cdots + a_k$$
$$\le a_1 + \cdots + a_N + c_{N+1} + \cdots + c_k$$
$$\le a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n$$

(b) Since all of the a_n are nonnegative, the partial sums of the series form a nondecreasing sequence of real numbers. Part (a) shows that the sequence is bounded above, so it must converge to a limit.

43. **(a)** For $k \le N$, it's obvious that

$$d_1 + \cdots + d_k \le d_1 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n.$$

For all $k > N$,

$$d_1 + \cdots + d_k = d_1 + \cdots + d_N + d_{N+1} + \cdots + d_k$$
$$\le d_1 + \cdots + d_N + a_{N+1} + \cdots + a_k$$
$$\le d_1 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n$$

(b) If $\sum a_n$ converged, that would imply that $\sum d_n$ was also convergent.

44. Answers will vary.

45. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$

$s_1 = 1 - \frac{1}{5}$

$s_2 = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) = 1 - \frac{1}{9}$

$s_3 = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) = 1 - \frac{1}{13}$

$s_n = 1 - \frac{1}{4n+1}$

$S = \lim_{n\to\infty} s_n = 1$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty}\left(\frac{3}{2n-1} - \frac{3}{2n+1}\right)$

$s_1 = 3 - \frac{3}{3}$

$s_2 = (3 - 1) + \left(1 - \frac{3}{5}\right) = 3 - \frac{3}{5}$

$s_3 = (3 - 1) + \left(1 - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{7}\right) = 3 - \frac{3}{7}$

$s_n = 3 - \frac{3}{2n+1}$

$S = \lim_{n\to\infty} s_n = 3$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2} = \sum_{n=1}^{\infty}\left[\frac{5}{(2n-1)^2} - \frac{5}{(2n+1)^2}\right]$

$s_1 = 5 - \frac{5}{9}$

$s_2 = \left(5 - \frac{5}{9}\right) + \left(\frac{5}{9} - \frac{5}{25}\right) = 5 - \frac{5}{25}$

$s_3 = \left(5 - \frac{5}{9}\right) + \left(\frac{5}{9} - \frac{5}{25}\right) + \left(\frac{5}{25} - \frac{5}{49}\right) = 5 - \frac{5}{49}$

$s_n = 5 - \frac{5}{(2n+1)^2}$

$S = \lim_{n\to\infty} s_n = 5$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$s_1 = 1 - \frac{1}{4}$

$s_2 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) = 1 - \frac{1}{9}$

$s_3 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) = 1 - \frac{1}{16}$

$s_n = 1 - \frac{1}{(n+1)^2}$

$S = \lim_{n\to\infty} s_n = 1$

49. $s_1 = 1 - \frac{1}{\sqrt{2}}$

$s_2 = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\sqrt{3}}$

$s_3 = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) = 1 - \frac{1}{\sqrt{4}}$

$s_n = 1 - \frac{1}{\sqrt{n+1}}$

$S = \lim_{n\to\infty} s_n = 1$

50. $s_1 = \frac{1}{\ln 3} - \frac{1}{\ln 2}$

$s_2 = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) = \frac{1}{\ln 4} - \frac{1}{\ln 2}$

$s_3 = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4}\right)$
$= \frac{1}{\ln 5} - \frac{1}{\ln 2}$

$s_n = \frac{1}{\ln(n+2)} - \frac{1}{\ln 2}$

$S = \lim_{n\to\infty} s_n = -\frac{1}{\ln 2}$

51. $s_1 = \tan^{-1} 1 - \tan^{-1} 2 = \frac{\pi}{4} - \tan^{-1} 2$

$s_2 = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3)$

$= \frac{\pi}{4} - \tan^{-1} 3$

$s_3 = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3) + (\tan^{-1} 3 - \tan^{-1} 4)$

$= \frac{\pi}{4} - \tan^{-1} 4$

$s_n = \frac{\pi}{4} - \tan^{-1} (n + 1)$

$S = \lim_{n\to\infty} s_n = \frac{\pi}{4} - \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

52. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Differentiate:

$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$

Multiply by x:

$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$

Differentiate:

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{(1-x)^2(1) - (x)(2)(1-x)(-1)}{(1-x)^4}$$

$$= \frac{(1-x) + 2x}{(1-x)^3}$$

$$= \frac{x+1}{(1-x)^3}$$

$\frac{x+1}{(1-x)^3} = \sum_{n=0}^{\infty} n^2x^{n-1}$

Multiply by x:

$\frac{x(x+1)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2x^n$

Let $x = \frac{1}{2}$:

$$\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3} = \sum_{n=0}^{\infty} n^2\left(\frac{1}{2}\right)^n$$

$$6 = \sum_{n=0}^{\infty} \frac{n^2}{2^n}$$

The sum is 6.

■ Section 9.5 Testing Convergence at Endpoints (pp. 496–508)

Exploration 1 The p-Series Test

1. We first note that the Integral Test applies to any series of the form $\sum \frac{1}{n^p}$ where p is positive. This is because the function $f(x) = x^{-p}$ is continuous and positive for all $x > 0$, and $f'(x) = -p \cdot x^{p-1}$ is negative for all $x > 0$.

If $p > 1$:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{k\to\infty} \int_1^k \frac{1}{x^p}\,dx = \lim_{k\to\infty}\left(\left.\frac{x^{-p+1}}{-p+1}\right]_1^k\right)$$

$$= \lim_{k\to\infty}\left(\frac{1}{1-p}\cdot\left(\frac{1}{k^{p-1}} - 1\right)\right)$$

$$= 0 + \frac{1}{p-1} \text{ (since } p-1 > 0)$$

$$= \frac{1}{p-1} < \infty.$$

The series converges by the Intergral Test.

2. If $0 < p < 1$:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{k\to\infty} \int_1^k \frac{1}{x^p}\,dx$$

$$= \lim_{k\to\infty}\left(\left.\frac{x^{-p+1}}{-p+1}\right]_1^k\right)$$

$$= \lim_{k\to\infty}\left(\frac{1}{1-p}\cdot(k^{1-p} - 1)\right)$$

$$= \infty \text{ (since } 1 - p > 0).$$

The series diverges by the Integral Test.
If $p \le 0$, the series diverges by the nth Term Test. This completes the proof for $p < 1$.

3. If $p = 1$:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{k\to\infty} \int_1^k \frac{1}{x}\,dx$$

$$= \lim_{k\to\infty}\left(\left.\ln x\right]_1^k\right)$$

$$= \lim_{k\to\infty} \ln k = \infty.$$

The series diverges by the Integral Test.

Exploration 2 The Maclaurin Series of a Strange Function

1. Since $f^{(n)}(0) = 0$ for all n, the Maclaurin Series for f has all zero coefficients! The series is simply $\sum_{n=0}^{\infty} 0 \cdot x^n = 0$.
2. The series converges (to 0) for all values of x.
3. Since $f(x) = 0$ only at $x = 0$, the only place that this series actually converges to its f-value is at $x = 0$.

Quick Review 9.5

1. Converges, since it is of the form $\int_1^{\infty} \frac{1}{x^p}\,dx$ with $p > 1$
2. Diverges, limit comparison test with integral of $\frac{1}{x}$
3. Diverges, comparison test with integral of $\frac{1}{x}$
4. Converges, comparison test with integral of $\frac{2}{x^2}$
5. Diverges, limit comparison test with integral of $\frac{1}{\sqrt{x}}$
6. Yes, for $N = 0$
7. Yes, for $N = 2\sqrt{2}$
8. No, neither positive nor decreasing for $x > \sqrt{3}$
9. No, oscillates
10. No, not positive for $x \ge 1$

Section 9.5 Exercises

1. Diverges by the Integral Test, since $\int_1^{\infty} \frac{5}{x+1}\,dx$ diverges.
2. Diverges because $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, which diverges by the p-series Test.
3. Diverges by the Direct Comparison Test, since $\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 2$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges.
4. Diverges by the Integral Test, since $\int_1^{\infty} \frac{1}{2x-1}\,dx$ diverges.
5. Diverges, since it is a geometric series with $r = \frac{1}{\ln 2} \approx 1.44$.
6. Converges, since it is a geometric series with $r = \frac{1}{\ln 3} \approx 0.91$.
7. Diverges by the nth-Term Test, since $\lim_{n\to\infty} n \sin \frac{1}{n} = 1$.
8. Converges by the Direct Comparison Test, since $\frac{e^n}{1+e^{2n}} < e^{-n}$ for $n \ge 0$, and $\sum_{n=0}^{\infty} e^{-n}$ converges as a geometric series with $r = e^{-1} \approx 0.37$.
9. Converges by the Direct Comparison Test, since $\frac{\sqrt{n}}{n^2+1} < \frac{1}{n^{3/2}}$ for $n \ge 1$, and $\sum_{n=0}^{\infty} \frac{1}{n^{3/2}}$ converges as a p-series with $p = \frac{3}{2}$.
10. Converges by the Limit Comparison Test, since
$$\lim_{n\to\infty} \frac{\frac{5n^3 - 3n}{n^2(n+2)(n^2+5)}}{\frac{1}{n^2}} = 1, \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$
converges as a p-series with $p = 2$.
11. Diverges by the nth-Term Test, since
$$\lim_{n\to\infty} \frac{3^{n-1}+1}{3^n} = \frac{1}{3} \ne 0.$$
12. Converges by the Alternating Series Test. If $u_n = \frac{1}{\ln n}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$.
13. Diverges by the nth-Term Test, since $\lim_{n\to\infty} \frac{10^n}{n^{10}} = \infty$.
14. Converges by the Alternating Series Test. If $u_n = \frac{\sqrt{n}+1}{n+1}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$. (To show that u_n is decreasing, let $f(x) = \frac{\sqrt{x}+1}{x+1}$ and observe that
$$f'(x) = \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}+1)(1)}{(x+1)^2} = \frac{1 - x - 2\sqrt{x}}{2(x+1)^2\sqrt{x}},$$
which is negative, at least for $x \ge 1$.)
15. Diverges by the nth-Term Test since $\frac{\ln n}{\ln n^2} = \frac{\ln n}{2 \ln n} = \frac{1}{2}$, which means each term is $\pm\frac{1}{2}$.
16. Diverges by the Limit Comparison Test.
Let $a_n = \frac{1}{n} - \frac{1}{n^2}$ and $b_n = \frac{1}{n}$.
Then $a_n > 0$ and $b_n > 0$ for $n \ge 2$ and
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$
Since $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ also diverges.
17. Converges absolutely, because, absolutely, it is a geometric series with $r = 0.1$.
18. Converges conditionally:
If $u_n = \frac{1+n}{n^2} = \frac{1}{n} + \frac{1}{n^2}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$, so $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1+n}{n^2}$ converges by the Alternating Series Test.

But $\sum_{n=1}^{\infty} \frac{1+n}{n^2}$ diverges by the Direct Comparison Test, since $\frac{1+n}{n^2} \ge \frac{1}{n}$ for $n \ge 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

19. Converges absolutely, since $\sum_{n=1}^{\infty} n^2\left(\frac{2}{3}\right)^n$ converges by the Ratio Test:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = (n+1)^2\left(\frac{2}{3}\right)^{n+1} \cdot \frac{1}{n^2}\left(\frac{3}{2}\right)^n = \frac{2}{3} < 1.$

20. Converges conditionally.

If $u_n = \frac{1}{n \ln n}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$, so $\sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{n \ln n}$ converges by the Alternating Series Test.

But $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test, since $\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty}\Big[\ln|\ln x|\Big]_2^b = \infty.$

21. Diverges by the nth-Term Test, since $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$ and so the terms do not approach 0.

22. Converges absolutely, since $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|$ converges by direct comparison to $\sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges as a p-series with $p = 2$.

23. Converges conditionally:

If $u_n = \frac{1}{1+\sqrt{n}}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$ converges by the Alternating Series Test.

But $\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$ diverges by direct comparison to $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$, which diverges as a p-series with $p = \frac{1}{2}$.

24. Converges absolutely, since $\sum_{n=1}^{\infty}\left|\frac{\cos n\pi}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which converges as a p-series.

25. Converges conditionally, since $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n} = -\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. (See Examples 2 and 4.)

26. Converges conditionally:

If $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges by the Alternating Series Test.

But $\sum_{n=1}^{\infty} u_n$ diverges by the Limit Comparison Test:

If $v_n = \frac{1}{n^{1/2}}$, then $\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^{1/2}}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{2}$.

Since $\sum_{n=1}^{\infty} v_n$ diverges as a p-series with $p = \frac{1}{2}$, $\sum_{n=1}^{\infty} u_n$ also diverges.

27. This is a geometric series which converges only for $|x| < 1$.

(a) $(-1, 1)$

(b) $(-1, 1)$

(c) None

28. This is a geometric series which converges only for $|x+5| < 1$, or $-6 < x < -4$.

(a) $(-6, -4)$

(b) $(-6, -4)$

(c) None

29. This is a geometric series which converges only for $|4x+1| < 1$, or $-\frac{1}{2} < x < 0$.

(a) $\left(-\frac{1}{2}, 0\right)$

(b) $\left(-\frac{1}{2}, 0\right)$

(c) None

30. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|3x-2|^{n+1}}{n+1}\cdot\frac{n}{|3x-2|^n} = |3x-2|$

The series converges absolutely when $|3x-2| < 1$, or $\frac{1}{3} < x < 1$. Check $x = \frac{1}{3}$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges conditionally. Check $x = 1$: $\sum_{n=1}^{\infty}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

(a) $\left[\frac{1}{3}, 1\right)$

(b) $\left(\frac{1}{3}, 1\right)$

(c) At $x = \frac{1}{3}$

31. This is a geometric series which converges only for $\left|\frac{x-2}{10}\right| < 1$, or $-8 < x < 12$.

(a) $(-8, 12)$

(b) $(-8, 12)$

(c) None

32. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{n+3}\cdot\frac{n+2}{n|x|^n} = |x|$

The series converges absolutely when $|x| < 1$, or $-1 < x < 1$. For $|x| \geq 1$, the series diverges by the nth-Term Test.

(a) $(-1, 1)$

(b) $(-1, 1)$

(c) None

33. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)\sqrt{n+1}\cdot 3^{n+1}}\cdot\frac{n\sqrt{n}\,3^n}{|x|^n} = \frac{|x|}{3}$

The series converges absolutely for $|x| < 3$. Furthermore, when $|x| = 3$, $\sum_{n=1}^{\infty}\left|\frac{x^n}{n\sqrt{n}\,3^n}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which also converges as a p-series with $p = \frac{3}{2}$.

(a) $[-3, 3]$
(b) $[-3, 3]$
(c) None

34. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{2n+3}}{(n+1)!}\cdot\frac{n!}{|x|^{2n+1}} = \lim_{n\to\infty}\frac{x^2}{n+1} = 0$

The series converges absolutely for all real numbers.

(a) All real numbers
(b) All real numbers
(c) None

35. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x+3|^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n|x+3|^n} = \frac{|x+3|}{5}$

The series converges absolutely for $\frac{|x+3|}{5} < 1$, or $-8 < x < 2$. For $\frac{|x+3|}{5} \geq 1$, the series diverges by the nth-Term Test.

(a) $(-8, 2)$
(b) $(-8, 2)$
(c) None

36. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{4^{n+1}[(n+1)^2+1]}\cdot\frac{4^n(n^2+1)}{n|x|^n} = \frac{|x|}{4}$

The series converges absolutely for $\frac{|x|}{4} < 1$, or $-4 < x < 4$.

Check $x = -4$:

$\sum_{n=0}^{\infty}\frac{(-1)^n n}{n^2+1}$ converges by the Alternating Series Test.

Check $x = 4$:

$\sum_{n=0}^{\infty}\frac{n}{n^2+1}$ diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n}$.

(a) $[-4, 4)$
(b) $(-4, 4)$
(c) At $x = -4$

37. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}\,|x|^{n+1}}{3^{n+1}}\cdot\frac{3^n}{\sqrt{n}\,|x|^n} = \frac{|x|}{3}$

The series converges absolutely for $|x| < 3$, or $-3 < x < 3$.

For $|x| \geq 3$, the series diverges by the nth-Term Test.

(a) $(-3, 3)$
(b) $(-3, 3)$
(c) None

38. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x-4|^{n+1}}{n!|x-4|^n}$

$= \lim_{n\to\infty}(n+1)|x-4|$

$= \begin{cases} 0, & x = 4 \\ \infty, & x \neq 4 \end{cases}$

(a) Only at $x = 4$
(b) At $x = 4$
(c) None

39. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|-2|^{n+1}(n+2)|x-1|^{n+1}}{|-2|^n(n+1)|x-1|^n} = |2(x-1)|$

The series converges absolutely for $|2(x-1)| < 1$, or $\frac{1}{2} < x < \frac{3}{2}$. For $|2(x-1)| \geq 1$, the series diverges by the nth-Term Test.

(a) $\left(\frac{1}{2}, \frac{3}{2}\right)$
(b) $\left(\frac{1}{2}, \frac{3}{2}\right)$
(c) None

40. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|4x-5|^{2n+3}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{|4x-5|^{2n+1}} = (4x-5)^2$

The series converges absolutely for $(4x-5)^2 < 1$, or $1 < x < \frac{3}{2}$. Check $x = 1$: $\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^{3/2}} = -\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges as a p-series with $p = \frac{3}{2}$. Check $x = \frac{3}{2}$: $\sum_{n=1}^{\infty}\frac{1^{2n+1}}{n^{3/2}}$ converges as a p-series with $p = \frac{3}{2}$.

(a) $\left[1, \frac{3}{2}\right]$
(b) $\left[1, \frac{3}{2}\right]$
(c) None

41. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x+\pi|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x+\pi|^n} = |x+\pi|$

The series converges absolutely for $|x+\pi| < 1$, or $-\pi - 1 < x < -\pi + 1$.

Check $x = -\pi - 1$:

$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by Alternating Series Test.

Check $x = -\pi + 1$:

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges as a p-series with $p = \frac{1}{2}$.

(a) $[-\pi - 1, -\pi + 1)$
(b) $(-\pi - 1, -\pi + 1)$
(c) At $x = -\pi - 1$

42. This is a geometric series which converges only for

$|\ln x| < 1$, or $\frac{1}{e} < x < e$.

(a) $\left(\frac{1}{e}, e\right)$

(b) $\left(\frac{1}{e}, e\right)$

(c) None

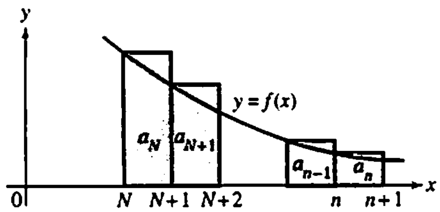
43. $n = 13 \times 10^9 \cdot 365 \cdot 24 \cdot 3600 = 4.09968 \times 10^{17}$
$\ln(n + 1) < \text{sum} < 1 + \ln n$
$\ln(4.09968 \times 10^{17} + 1) < \text{sum} < 1 + \ln(4.09968 \times 10^{17})$
$40.5548... < \text{sum} < 41.5548...$
$40.554 < \text{sum} < 41.555$

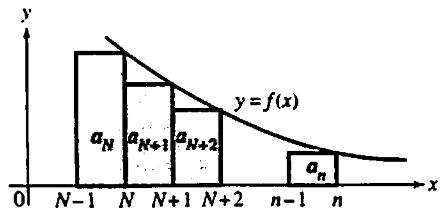
44. Comparing areas in the figures, we have for all $n \geq 1$,

$$\int_1^{n+1} f(x)\,dx < a_1 + \cdots + a_n < a_1 + \int_1^n f(x)\,dx.$$

If the integral diverges, it must go to infinity, and the first inequality forces the partial sums of the series to go to infinity as well, so the series is divergent. If the integral converges, then the second inequality puts an upper bound on the partial sums of the series, and since they are a nondecreasing sequence, they must converge to a finite sum for the series. (See the explanation preceding Exercise 42 in Section 9.4.)

45.





Comparing areas in the figures, we have for all $n \geq N$,

$$\int_N^{n+1} f(x)\,dx < a_N + \cdots + a_n < a_N + \int_N^n f(x)\,dx.$$

If the integral diverges, it must go to infinity, and the first inequality forces the partial sums of the series to go to infinity as well, so the series is divergent. If the integral converges, then the second inequality puts an upper bound on the partial sums of the series, and since they are a nondecreasing sequence, they must converge to a finite sum for the series. (See the explanation preceding Exercise 42 in Section 9.4.)

46. (a) Diverges by the Limit Comparison Test.

Let $a_k = \frac{1}{\sqrt{2k + 7}}$ and $b_k = \frac{1}{k^{1/2}}$. Then $a_k > 0$ and $b_k > 0$ for $k \geq 1$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^{1/2}}{\sqrt{2k + 7}} = \frac{1}{\sqrt{2}}$. Since $\sum_{k=1}^{\infty} b_k$ diverges as a p-series with $p = \frac{1}{2}$, $\sum_{k=1}^{\infty} a_k$ also diverges.

(b) Diverges by the nth-Term Test, since

$$\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e \neq 0.$$

(c) Converges absolutely by the Comparison Test, since

$\left|\frac{\cos k}{k^2 + \sqrt{k}}\right| < \frac{1}{k^2}$ for $k \geq 1$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges as a p-series with $p = 2$.

(d) Diverges by the integral test, since

$$\int_3^{\infty} \frac{18}{x \ln x}\,dx = \lim_{b\to\infty} \left[18 \ln |\ln x|\right]_3^b = \infty$$

47. One possible answer: $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

This series diverges by the integral test, since

$\int_3^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \left[\ln |\ln x|\right]_3^b = \infty$. Its partial sums are roughly $\ln(\ln n)$, so they are much smaller than the partial sums for the harmonic series, which are about $\ln n$.

48. (a) $a_k = (-1)^{k+1} \int_0^{1/k} 6(kx)^2\,dx$

$$= (-1)^{k+1}\left[2k^2x^3\right]_0^{1/k}$$

$$= (-1)^{k+1}\left(\frac{2}{k}\right)$$

(b) The series converges by the Alternating Series Test.

(c) The first few partial sums are:

$S_1 = 2, S_2 = 1, S_3 = \frac{5}{3}, S_4 = \frac{7}{6}, S_5 = \frac{47}{30}, S_6 = \frac{37}{30}, S_7 = \frac{319}{210}, S_8 = \frac{533}{420}, S_9 = \frac{1879}{1260}$. For an alternating series, the sum is between any two adjacent partial sums, so $1 < S_8 \leq \text{sum} \leq S_9 < \frac{3}{2}$.

49. (a) Diverges by the Limit Comparison Test. Let

$a_n = \frac{n}{3n^2 + 1}$ and $b_n = \frac{1}{n}$. Then $a_n > 0$ and $b_n > 0$ for $n \geq 1$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{3n^2 + 1} = \frac{1}{3}$. Since $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ diverges.

(b) $S = \sum_{n=1}^{\infty} \frac{n}{3n^2+1} \cdot \frac{3}{n} = \sum_{n=1}^{\infty} \frac{3}{3n^2+1}$.
This series converges by the Direct Comparison Test, since $\frac{3}{3n^2+1} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as a p-series with $p = 2$.

50. (a) From the list of Maclaurin series in Section 9.2,
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$.

(b) $-1 < x \le 1$

(c) To estimate $\ln \frac{3}{2}$, we would let $x = \frac{1}{2}$

The truncation error is less than the magnitude of the sixth nonzero term, or

$\left|-\frac{x^6}{6}\right| = \frac{1}{2^6 \cdot 6} = \frac{1}{384} < 0.002605$

Thus, a bound for the (absolute) truncation error is 0.002605.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x^2)^n}{n} = \frac{1}{2}\ln(1+x^2)$

51. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{2^{k+1}|x|^{k+1}}{\ln(k+3)} \cdot \frac{\ln(k+2)}{2^k|x|^k} = 2|x|$
The series converges absolutely for $|x| < \frac{1}{2}$,
or $-\frac{1}{2} < x < \frac{1}{2}$.
Check $x = -\frac{1}{2}$:
$\sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}$ converges by the Alternating Series Test.
Check $x = \frac{1}{2}$:
$\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$ diverges by the Direct Comparison Test, since $\frac{1}{\ln(k+2)} > \frac{1}{k}$ for $k \ge 2$ and $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges. The original series converges for $-\frac{1}{2} \le x < \frac{1}{2}$.

52. (a) The series converges by the Direct Comparison Test, since $\frac{1}{n^p \ln n} < \frac{1}{n^p}$ for $n \ge 3$, and $\sum_{n=3}^{\infty} \frac{1}{n^p}$ converges as a p-series when $p > 1$.

(b) For $p = 1$, the series is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$, which diverges by the Integral Test, since $\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty}\Big[\ln(\ln x)\Big]_2^b = \infty$.

(c) For $0 \le p < 1$, we have $\frac{1}{n^p \ln n} > \frac{1}{n \ln n}$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges by the Direct Comparison Test with $\frac{1}{n \ln n}$ from part (b).

53. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$, so at $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. This series converges by the Alternating Series Test.

54. $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
At $x = -1$, the sequence is
$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test. At $x = 1$, the sequence is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test.

55. (a) It fails to satisfy $u_n \ge u_{n+1}$ for all $n \ge N$.

(b) The sum is $\left(\sum_{n=1}^{\infty} \frac{1}{3^n}\right) - \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) = \frac{1/3}{1-1/3} - \frac{1/2}{1-1/2}$
$= \frac{1}{2} - 1$
$= -\frac{1}{2}$.

56. Answers will vary.

57. (a) $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{1}{2}$
The series converges.

(b) $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n-1}\right)^n} = \lim_{n\to\infty}\frac{n}{2n-1} = \frac{1}{2}$
The series converges.

(c) $\lim_{n\to\infty,\, n \text{ odd}} \sqrt[n]{a_n} = \lim_{n\to\infty,\, n \text{ odd}} \sqrt[n]{\frac{n}{2^n}}$
$= \lim_{n\to\infty,\, n \text{ odd}} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}$

$\lim_{n\to\infty,\, n \text{ even}} \sqrt[n]{a_n} = \lim_{n\to\infty,\, n \text{ even}} \sqrt[n]{\frac{1}{2^n}} = \frac{1}{2}$

Thus, $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$, so the series converges.

58. (a) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{|x-1|^n}{4^n}} = \frac{|x-1|}{4}$
The series converges absolutely if $\frac{|x-1|}{4} < 1$, or $-3 < x < 5$.

Check $x = -3$: $\sum_{n=0}^{\infty} (-1)^n$ diverges.

Check $x = 5$: $\sum_{n=0}^{\infty} 1^n$ diverges.

The interval of convergence is $(-3, 5)$.

58. continued

(b) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{|x-2|^n}{n \cdot 3^n}} = \lim_{n\to\infty} \frac{|x-2|}{\sqrt[n]{n} \cdot 3} = \frac{|x-2|}{3}$

The series converges absolutely if $\frac{|x-2|}{3} < 1$, or

$-1 < x < 5$.

Check $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Check $x = 5$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

The interval of convergence is $[-1, 5)$.

(c) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{2^n|x|^n} = 2|x|$

The series converges absolutely if

$2|x| < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$.

Check $x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} (-1)^n$ diverges.

Check $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} 1$ diverges.

The interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(d) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|\ln x|^n} = |\ln x|$

The series converges absolutely if $|\ln x| < 1$, or

$\frac{1}{e} < x < e$.

Check: $x = \frac{1}{e}$: $\sum_{n=0}^{\infty} \left(\ln \frac{1}{e}\right)^n = \sum_{n=0}^{\infty} (-1)^n$ diverges.

Check $x = e$: $\sum_{n=0}^{\infty} (\ln e)^n = \sum_{n=0}^{\infty} 1^n$ diverges.

The interval of convergence is $\left(\frac{1}{e}, e\right)$.

■ Chapter 9 Review Exercises
(pp. 509–511)

1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|-x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|-x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$

The series converges absolutely for all x.

(a) ∞
(b) All real numbers
(c) All real numbers
(d) None

2. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x+4|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|x+4|^n} = \frac{|x+4|}{3}$

The series converges absolutely for $\frac{|x+4|}{3} < 1$,

or $-7 < x < -1$.

Check $x = -7$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

Check $x = -1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(a) 3
(b) $[-7, -1)$
(c) $(-7, -1)$
(d) At $x = -7$

3. This is a geometric series, so it converges absolutely when $|r| < 1$ and diverges for all other values of x. Since $r = \frac{2}{3}(x-1)$, the series converges absolutely when $\left|\frac{2}{3}(x-1)\right| < 1$, or $-\frac{1}{2} < x < \frac{5}{2}$.

(a) $\frac{3}{2}$
(b) $\left(-\frac{1}{2}, \frac{5}{2}\right)$
(c) $\left(-\frac{1}{2}, \frac{5}{2}\right)$
(d) None

4. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{|x-1|^{2n-2}}$
$= \lim_{n\to\infty} \frac{|x-1|^2}{(2n+1)(2n)} = 0$

The series converges absolutely for all x.

(a) ∞
(b) All real numbers
(c) All real numbers
(d) None

5. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|3x-1|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|3x-1|^n} = |3x-1|$

The series converges absolutely for $|3x-1| < 1$, or $0 < x < \frac{2}{3}$. Furthermore, when $|3x-1| = 1$, we have $|a_n| = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2$, so $\sum_{n=1}^{\infty} a_n$ also converges absolutely at the interval endpoints.

(a) $\frac{1}{3}$
(b) $\left[0, \frac{2}{3}\right]$
(c) $\left[0, \frac{2}{3}\right]$
(d) None

6. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)|x|^{3n+3}}{(n+1)|x|^{3n}} = |x|^3$

The series converges absolutely for $|x|^3 < 1$, or $-1 < x < 1$. When $|x| \ge 1$, the series diverges by the nth Term Test.

(a) 1
(b) $(-1, 1)$
(c) $(-1, 1)$
(d) None

7. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)|2x+1|^{n+1}}{(2n+3)2^{n+1}}\cdot\frac{(2n+1)2^n}{(n+1)|2x+1|^n}$
$= \frac{|2x+1|}{2}$

The series converges absolutely for $\frac{|2x+1|}{2} < 1$, or $-\frac{3}{2} < x < \frac{1}{2}$. When $\frac{|2x+1|}{2} \ge 1$, the series diverges by the nth-Term Test.

(a) 1
(b) $\left(-\frac{3}{2}, \frac{1}{2}\right)$
(c) $\left(-\frac{3}{2}, \frac{1}{2}\right)$
(d) None

8. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{|x|^n} = |x|\lim_{n\to\infty}\frac{n^n}{(n+1)(n+1)^n}$
$= |x|\lim_{n\to\infty}\frac{1}{(n+1)\left(1+\frac{1}{n}\right)^n} = \frac{|x|}{e}\lim_{n\to\infty}\frac{1}{n+1} = 0$

The series converges absolutely for all x.

Another way to see that the series must converge is to observe that for $n \ge 2x$, we have $\left|\frac{x^n}{n^n}\right| \le \left(\frac{1}{2}\right)^n$, so the terms are (eventually) bounded by the terms of a convergent geometric series.

A third way to solve this exercise is to use the nth Root Test (see Exercises 57–58 in Section 9.5).

(a) ∞
(b) All real numbers
(c) All real numbers
(d) None

9. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x|^n} = |x|$

The series converges absolutely for $|x| < 1$, or $-1 < x < 1$.

Check $x = -1$:
$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by the Alternating Series Test.

Check $x = 1$:
$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges as a p-series with $p = \frac{1}{2}$.

(a) 1
(b) $[-1, 1)$
(c) $(-1, 1)$
(d) At $x = -1$

10. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}|x|^{n+1}}{(n+1)^e}\cdot\frac{n^e}{e^n|x|^n} = e|x|$

The series converges absolutely for $e|x| < 1$, or $-\frac{1}{e} < x < \frac{1}{e}$.

Furthermore, when $e|x| = 1$, we have $|a_n| = \frac{1}{n^e}$ and $\sum_{n=1}^{\infty}\frac{1}{n^e}$ converges as a p-series with $p = e$, so $\sum_{n=1}^{\infty} a_n$ also converges absolutely at the interval endpoints.

(a) $\frac{1}{e}$
(b) $\left[-\frac{1}{e}, \frac{1}{e}\right]$
(c) $\left[-\frac{1}{e}, \frac{1}{e}\right]$
(d) None

11. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)|x|^{2n+1}}{3^{n+1}}\cdot\frac{3^n}{(n+1)|x|^{2n-1}} = \frac{x^2}{3}$

The series converges absolutely when $\frac{x^2}{3} < 1$, or $-\sqrt{3} < x < \sqrt{3}$.

When $|x| \ge \sqrt{3}$, the series diverges by the nth Term Test.

(a) $\sqrt{3}$
(b) $(-\sqrt{3}, \sqrt{3})$
(c) $(-\sqrt{3}, \sqrt{3})$
(d) None

12. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-1|^{2n+3}}{2n+3}\cdot\frac{2n+1}{|x-1|^{2n+1}} = |x-1|^2$

The series converges absolutely when $|x-1|^2 < 1$,

or $0 < x < 2$.

Check $x = 0$: $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n-1}}{2n+1} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges conditionally by the Alternating Series Test.

Check $x = 2$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges conditionally by the Alternating Series Test.

(a) 1

(b) $[0, 2]$

(c) $(0, 2)$

(d) At $x = 0$ and $x = 2$

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x|^{2n+2}}{2^{n+1}}\cdot\frac{2^n}{n!|x|^{2n}}$

$= \lim_{n\to\infty}\frac{(n+1)x^2}{2} = \begin{cases}0, & x = 0\\ \infty, & x \neq 0\end{cases}$

The series converges only at $x = 0$.

(a) 0

(b) $x = 0$ only

(c) $x = 0$

(d) None

14. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|10x|^{n+1}}{\ln(n+1)}\cdot\frac{\ln n}{|10x|^n} = |10x|$

The series converges absolutely for $|10x| < 1$,

or $-\frac{1}{10} < x < \frac{1}{10}$.

Check $n = -\frac{1}{10}$: $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test.

Check $n = \frac{1}{10}$: $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by the Direct Comparison Test, since $\frac{1}{\ln n} > \frac{1}{n}$ for $n \geq 2$ and $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges.

(a) $\frac{1}{10}$

(b) $\left[-\frac{1}{10}, \frac{1}{10}\right)$

(c) $\left(-\frac{1}{10}, \frac{1}{10}\right)$

(d) At $x = -\frac{1}{10}$

15. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)!|x|^{n+1}}{(n+1)!|x|^n} = \lim_{n\to\infty}(n+2)|x| = \infty\ (x \neq 0)$

The series converges only at $x = 0$.

(a) 0

(b) $x = 0$ only

(c) $x = 0$

(d) None

16. This is a geometric series with $r = \frac{x^2-1}{2}$, so it converges absolutely when $\left|\frac{x^2-1}{2}\right| < 1$, or $-\sqrt{3} < x < \sqrt{3}$. It diverges for all other values of x.

(a) $\sqrt{3}$

(b) $(-\sqrt{3}, \sqrt{3})$

(c) $(-\sqrt{3}, \sqrt{3})$

(d) None

17. $f(x) = \frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots$, evaluated at $x = \frac{1}{4}$. Sum $= \frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$.

18. $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}$, evaluated at $x = \frac{2}{3}$. Sum $= \ln\left(1+\frac{2}{3}\right) = \ln\left(\frac{5}{3}\right)$.

19. $f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$, evaluated at $x = \pi$. Sum $= \sin\pi = 0$.

20. $f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$, evaluated at $x = \frac{\pi}{3}$. Sum $= \cos\frac{\pi}{3} = \frac{1}{2}$.

21. $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$, evaluated at $x = \ln 2$. Sum $= e^{\ln 2} = 2$.

22. $f(x) = \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + \cdots$, evaluated at $x = \frac{1}{\sqrt{3}}$. Sum $= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. (Note that when n is replaced by $n-1$, the general term of $\tan^{-1}x$ becomes $(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$, which matches the general term given in the exercise.)

23. Replace x by $6x$ in the Maclaurin series for $\frac{1}{1-x}$ given at the end of Section 9.2.

$$\frac{1}{1-6x} = 1 + (6x) + (6x)^2 + \cdots + (6x)^n + \cdots$$
$$= 1 + 6x + 36x^2 + \cdots + (6x)^n + \cdots$$

24. Replace x by x^3 in the Maclaurin series for $\frac{1}{1+x}$ given at the end of Section 9.2.

$$\frac{1}{1+x^3} = 1 - (x^3) + (x^3)^2 - \cdots + (-x^3)^n + \cdots$$
$$= 1 - x^3 + x^6 - \cdots + (-1)^n x^{3n} + \cdots$$

25. The Maclaurin series for a polynomial is the polynomial itself: $1 - 2x^2 + x^9$.

26. $\frac{4x}{1-x} = 4x\left(\frac{1}{1-x}\right)$
$= 4x(1 + x + x^2 + \cdots + x^n + \cdots)$
$= 4x + 4x^2 + 4x^3 + \cdots + 4x^{n+1} + \cdots$

27. Replace x by πx in the Maclaurin series for $\sin x$ given at the end of Section 9.2.
$\sin \pi x = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots + (-1)^n\frac{(\pi x)^{2n+1}}{(2n+1)!} + \cdots$

28. Replace x by $\frac{2x}{3}$ in the Maclaurin series for $\sin x$ given at the end of Section 9.2
$-\sin\frac{2x}{3} = -\left(\frac{2x}{3} - \frac{\left(\frac{2x}{3}\right)^3}{3!} + \frac{\left(\frac{2x}{3}\right)^5}{5!} - \cdots + (-1)^n\frac{\left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!}\right)$
$= -\frac{2x}{3} + \frac{4x^3}{81} - \frac{4x^5}{3645} + \cdots + \frac{(-1)^{n+1}\left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!}$

29. $-x + \sin x = -x + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right.$
$\left. + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$
$= -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$

30. $\frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right)$
$+ \frac{1}{2}\left(1 - x + \frac{x^2}{2!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right)$
$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$

31. Replace x by $\sqrt{5x}$ in the Maclaurin series for $\cos x$ given at the end of Section 9.2.
$\cos\sqrt{5x} = 1 - \frac{(\sqrt{5x})^2}{2!} + \frac{(\sqrt{5x})^4}{4!} - \cdots$
$+ (-1)^n\frac{(\sqrt{5x})^{2n}}{(2n)!} + \cdots$
$= 1 - \frac{5x}{2!} + \frac{(5x)^2}{4!} - \cdots + (-1)^n\frac{(5x)^n}{(2n)!} + \cdots$

32. Replace x by $\frac{\pi x}{2}$ in the Maclaurin series for e^x given at the end of Section 9.2.
$e^{\pi x/2} = 1 + \frac{\pi x}{2} + \frac{\left(\frac{\pi x}{2}\right)^2}{2!} + \cdots + \frac{\left(\frac{\pi x}{2}\right)^n}{n!} + \cdots$
$= 1 + \frac{\pi x}{2} + \frac{\pi^2 x^2}{8} + \cdots + \frac{1}{n!}\left(\frac{\pi x}{2}\right)^n + \cdots$

33. Use the Maclaurin series for e^x given at the end of Section 9.2.
$xe^{-x^2} = x\left[1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots\right]$
$= x - x^3 + \frac{x^5}{2!} - \cdots + (-1)^n\frac{x^{2n+1}}{n!} + \cdots$

34. Replace x by $3x$ in the Maclaurin series for $\tan^{-1} x$ given at the end of Section 9.2.
$\tan^{-1} 3x = 3x - \frac{(3x)^3}{3} + \frac{(3x)^5}{5} - \cdots + (-1)^n\frac{(3x)^{2n+1}}{2n+1} + \cdots$

35. Replace x by $-2x$ in the Maclaurin series for $\ln(1 + x)$ given at the end of Section 9.2.
$\ln(1 - 2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \cdots$
$+ (-1)^{n-1}\frac{(-2x)^n}{n} + \cdots$
$= -2x - 2x^2 - \frac{8x^3}{3} - \cdots - \frac{(2x)^n}{n} - \cdots.$

36. Use the Maclaurin series for $\ln(1 + x)$ given at the end of Section 9.2.
$x\ln(1 - x) = x\ln[1 + (-x)]$
$= x\left[-x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \cdots + (-1)^{n-1}\frac{(-x)^n}{n} + \cdots\right]$
$= -x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \cdots - \frac{x^{n+1}}{n} - \cdots$

37. $f(2) = (3 - x)^{-1}\Big|_{x=2} = 1$
$f'(2) = (3 - x)^{-2}\Big|_{x=2} = 1$
$f''(2) = 2(3 - x)^{-3}\Big|_{x=2} = 2$, so $\frac{f''(2)}{2!} = 1$
$f'''(2) = 6(3 - x)^{-4}\Big|_{x=2} = 6$, so $\frac{f'''(2)}{3!} = 1$
$f^{(n)}(2) = n!(3 - x)^{-n-1}\Big|_{x=2} = n!$, so $\frac{f^{(n)}(2)}{n!} = 1$
$\frac{1}{3-x} = 1 + (x - 2) + (x - 2)^2 + (x - 2)^3 + \cdots$
$+ (x - 2)^n + \cdots$

38. $f(-1) = (x^3 - 2x^2 + 5)\Big|_{x=-1} = 2$
$f'(-1) = (3x^2 - 4x)\Big|_{x=-1} = 7$
$f''(-1) = (6x - 4)\Big|_{x=-1} = -10$, so $\frac{f''(-1)}{2!} = -5$
$f'''(-1) = 6\Big|_{x=-1} = 6$, so $\frac{f'''(-1)}{3!} = 1$
$f^{(n)}(-1) = 0$ for $n \geq 4$.
$x^3 - 2x^2 + 5 = 2 + 7(x + 1) - 5(x + 1)^2 + (x + 1)^3$
This is a finite series and the general term for $n \geq 4$ is 0.

39. $f(3) = \frac{1}{x}\Big|_{x=3} = \frac{1}{3}$

$f'(3) = -x^{-2}\Big|_{x=3} = -\frac{1}{9}$

$f''(3) = 2x^{-3}\Big|_{x=3} = \frac{2}{27}$, so $\frac{f''(3)}{2!} = \frac{1}{27}$

$f'''(3) = -6x^{-4}\Big|_{x=3} = -\frac{2}{27}$, so $\frac{f'''(3)}{3!} = -\frac{1}{81}$

$\frac{f^{(n)}(3)}{n!} = \frac{(-1)^n}{3^{n+1}}$

$$\frac{1}{x} = \frac{1}{3} - \frac{1}{9}(x-3) + \frac{1}{27}(x-3)^2 - \frac{1}{81}(x-3)^3 + \cdots + (-1)^n\frac{(x-3)^n}{3^{n+1}}$$

40. $f(\pi) = \sin x\Big|_{x=\pi} = 0$

$f'(\pi) = \cos x\Big|_{x=\pi} = -1$

$f''(\pi) = -\sin x\Big|_{x=\pi} = 0$, so $\frac{f''(\pi)}{2!} = 0$

$f'''(\pi) = -\cos x\Big|_{x=\pi} = 1$, so $\frac{f'''(\pi)}{3!} = \frac{1}{6}$

$$f^{(k)}(\pi) = \begin{cases} 0, & \text{if } k \text{ is even} \\ -1, & \text{if } k = 2n+1, n \text{ even} \\ 1, & \text{if } k = 2n+1, n \text{ odd} \end{cases}$$

$$\sin x = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 - \cdots + (-1)^{n+1}\frac{1}{(2n+1)!}(x-\pi)^{2n+1} + \cdots$$

41. Diverges, because it is -5 times the harmonic series:

$$\sum_{n=1}^{\infty}\frac{-5}{n} = -5\sum_{n=1}^{\infty}\frac{1}{n} = -\infty$$

42. Converges conditionally.

If $u_n = \frac{1}{\sqrt{n}}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. The convergence is conditional because $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series $\left(p = \frac{1}{2}\right)$.

43. Converges absolutely by the Direct Comparison Test, since $0 \le \frac{\ln n}{n^3} < \frac{1}{n^2}$ for $n \ge 1$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges as a p-series with $p = 2$.

44. Converges absolutely by the Ratio Test, since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+2}{(n+1)!}\cdot\frac{n!}{n+1} = \lim_{n\to\infty}\frac{n+2}{(n+1)^2} = 0.$$

45. Converges conditionally:

If $u_n = \frac{1}{\ln(n+1)}$, then $\{u_n\}$ is a decreasing sequence of positive terms with $\lim_{n\to\infty} u_n = 0$, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$ converges by the Alternating Series Test. The convergence is conditional because $\frac{1}{\ln(n+1)} > \frac{1}{n}$ for $n \ge 1$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so $\sum_{n=1}^{\infty}\frac{1}{\ln(n+1)}$ diverges by the Direct Comparison Test.

46. Converges absolutely by the Integral Test, because

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_2^b = \frac{1}{\ln 2}.$$

47. Converges absolutely the the Ratio Test, because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|-3|^{n+1}}{(n+1)!}\cdot\frac{n!}{|-3|^n} = \lim_{n\to\infty}\frac{3}{n+1} = 0.$$

48. Converges absolutely by the Direct Comparison Test, since $\frac{2^n3^n}{n^n} \le \left(\frac{1}{2}\right)^n$ for $n \ge 12$ and $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is a convergent geometric series. Alternately, we may use the Ratio Test or the nth-Root Test (see Exercise 57 and 58 in Section 9.5).

49. Diverges by the nth-Term Test, since

$\lim_{n\to\infty}\frac{(-1)^n(n^2+1)}{2n^2+n-1}$ does not exist.

50. Converges absolutely by the Direct Comparison Test, since

$\frac{1}{\sqrt{n(n+1)(n+2)}} < \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges as a p-series with $p = \frac{3}{2}$.

51. Converges absolutely by the Limit Comparison Test.

Let $a_n = \frac{1}{n\sqrt{n^2-1}}$ and $b_n = \frac{1}{n^2}$.

Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{n\sqrt{n^2-1}} = 1$ and $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges as a p-series ($p = 2$). Therefore $\sum_{n=2}^{\infty}a_n$ converges.

52. Diverges by the nth-Term Test, since

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e} \ne 0.$$

53. This is a telescoping series.

$$\sum_{n=3}^{\infty}\frac{1}{(2n-3)(2n-1)}=\sum_{n=3}^{\infty}\left(\frac{1}{2(2n-3)}-\frac{1}{2(2n-1)}\right)$$
$$s_1=\frac{1}{2(2\cdot3-3)}-\frac{1}{2(2\cdot3-1)}=\frac{1}{6}-\frac{1}{10}$$
$$s_2=\left(\frac{1}{6}-\frac{1}{10}\right)+\left(\frac{1}{10}-\frac{1}{14}\right)=\frac{1}{6}-\frac{1}{14}$$
$$s_3=\left(\frac{1}{6}-\frac{1}{10}\right)+\left(\frac{1}{10}-\frac{1}{14}\right)+\left(\frac{1}{14}-\frac{1}{18}\right)=\frac{1}{6}-\frac{1}{18}$$
$$s_n=\frac{1}{6}-\frac{1}{2(2n-1)}$$
$$S=\lim_{n\to\infty}s_n=\frac{1}{6}$$

54. This is a telescoping series.

$$\sum_{n=2}^{\infty}\frac{-2}{n(n+1)}=\sum_{n=2}^{\infty}\left(-\frac{2}{n}+\frac{2}{n+1}\right)$$
$$s_1=-\frac{2}{2}+\frac{2}{3}=-1+\frac{2}{3}$$
$$s_2=\left(-1+\frac{2}{3}\right)+\left(-\frac{2}{3}+\frac{2}{4}\right)=-1+\frac{2}{4}$$
$$s_3=\left(-1+\frac{2}{3}\right)+\left(-\frac{2}{3}+\frac{2}{4}\right)+\left(-\frac{2}{4}+\frac{2}{5}\right)=-1+\frac{2}{5}$$
$$s_n=-1+\frac{2}{n+2}$$
$$S=\lim_{n\to\infty}s_n=-1$$

55. (a) $P_3(x)=f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3$
$=1+4(x-3)+3(x-3)^2+2(x-3)^3$

$f(3.2)\approx P_3(3.2)=1.936$

(b) Since the Taylor series for f' can be obtained by term-by-term differentiation of the Taylor Series for f, the second order Taylor polynomial for f' at $x=3$ is $4+6(x-3)+6(x-3)^2$. Evaluated at $x=2.7$, $f'(2.7)\approx 2.74$.

(c) It underestimates the values, since $f''(3)=6$, which means the graph of f is concave up near $x=3$.

56. (a) Since the constant term is $f(4)$, $f(4)=7$. Since $-2=\frac{f'''(4)}{3!}$, $f'''(4)=-12$.

(b) Note that
$P_4'(x)=-3+10(x-4)-6(x-4)^2+24(x-4)^3$.
The second degree polynomial for f' at $x=4$ is given by the first three terms of this expression, namely $-3+10(x-4)-6(x-4)^2$. Evaluating at $x=4.3$, $f'(4.3)\approx-0.54$.

(c) The fourth order Taylor polynomial for $g(x)$ at $x=4$ is

$$\int_4^x[7-3(t-4)+5(t-4)^2-2(t-4)^3]\,dt$$
$$=\left[7t-\frac{3}{2}(t-4)^2+\frac{5}{3}(t-4)^3-\frac{1}{2}(t-4)^4\right]_4^x$$
$$=7(x-4)-\frac{3}{2}(x-4)^2+\frac{5}{3}(x-4)^3-\frac{1}{2}(x-4)^4$$

(d) No. One would need the entire Taylor series for $f(x)$, and it would have to converge to $f(x)$ at $x=3$.

57. (a) Use the Maclaurin series for $\sin x$ given at the end of Section 9.2.

$$5\sin\left(\frac{x}{2}\right)$$
$$=5\left[\frac{x}{2}-\frac{(x/2)^3}{3!}+\frac{(x/2)^5}{5!}-\cdots+(-1)^n\frac{(x/2)^{2n+1}}{(2n+1)!}+\cdots\right]$$
$$=\frac{5x}{2}-\frac{5x^3}{48}+\frac{x^5}{768}-\cdots+(-1)^n\frac{5}{(2n+1)!}\left(\frac{x}{2}\right)^{2n+1}+\cdots$$

(b) The series converges for all real numbers, according to the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{5}{(2n+3)!}\left|\frac{x}{2}\right|^{2n+3}\cdot\frac{(2n+1)!}{5}\left|\frac{2}{x}\right|^{2n+1}$$
$$=\lim_{n\to\infty}\frac{|x/2|^2}{(2n+3)(2n+2)}=0$$

(c) Note that the absolute value of $f^{(n)}(x)$ is bounded by $\frac{5}{2^n}$ for all x and all $n=1, 2, 3, \cdots$.

We may use the Remainder Estimation Theorem with $M=5$ and $r=\frac{1}{2}$.

So if $-2<x<2$, the truncation error using P_n is bounded by

$$\frac{5}{2^{n+1}}\cdot\frac{2^{n+1}}{(n+1)!}=\frac{5}{(n+1)!}.$$

To make this less than 0.1 requires $n\geq 4$. So, two terms (up through degree 4) are needed.

58. (a) Substitute $2x$ for x in the Maclaurin series for $\frac{1}{1-x}$ given at the end of Section 9.2.

$$\frac{1}{1-2x}=1+2x+(2x)^2+(2x)^3+\cdots+(2x)^n+\cdots$$
$$=1+2x+4x^2+8x^3+\cdots+(2x)^n+\cdots$$

(b) $\left(-\frac{1}{2},\frac{1}{2}\right)$. The series for $\frac{1}{1-t}$ is known to converge for $-1<t<1$, so by substituting $t=2x$, we find the resulting series converges for $-1<2x<1$.

58. continued

(c) $f\left(-\frac{1}{4}\right) = \frac{2}{3}$, so one percent is approximately 0.0067. It takes 7 terms (up through degree 6). This can be found by trial and error. Also, for $x = -\frac{1}{4}$, the series is the alternating series $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$. If you use the Alternating Series Estimation Theorem, it shows that 8 terms (up through degree 7) are sufficient since $\left|-\frac{1}{2}\right|^8 < 0.0067$. It is also a geometric series, and you could use the remainder formula for a geometric series to determine the number of terms needed. (See Example 2 in Section 9.3.)

59. (a)
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n n^n}$$
$$= \lim_{n\to\infty}\frac{|x|(n+1)^{n+1}}{(n+1)n^n}$$
$$= |x|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = |x|e$$

The series converges for $|x|e < 1$, or $|x| < \frac{1}{e}$, so the radius of convergence is $\frac{1}{e}$.

(b)
$$f\left(-\frac{1}{3}\right) \approx -\frac{1}{3}\cdot\frac{1}{1} + \left(-\frac{1}{3}\right)^2\cdot\frac{2^2}{2!} + \left(-\frac{1}{3}\right)^3\cdot\frac{3^3}{3!}$$
$$= -\frac{1}{3} + \frac{2}{9} - \frac{1}{6}$$
$$= -\frac{5}{18} \approx -0.278$$

(c) By the Alternating Series Estimation Theorem the error is no more than the magnitude of the next term, which is $\left|\left(-\frac{1}{3}\right)^4\cdot\frac{4^4}{4!}\right| = \frac{32}{243} \approx 0.132$.

60. (a)
$$f(3) = (x-2)^{-1}\Big|_{x=3} = 1$$
$$f'(3) = -(x-2)^{-2}\Big|_{x=3} = -1$$
$$f''(3) = 2(x-2)^{-3}\Big|_{x=3} = 2, \text{ so } \frac{f''(3)}{2!} = 1$$
$$f'''(3) = -6(x-2)^{-4}\Big|_{x=3} = -6, \text{ so } \frac{f'''(3)}{3!} = -1$$
$$f^{(n)}(3) = (-1)^n n!, \text{ so } \frac{f^{(n)}(3)}{n!} = (-1)^n$$
$$f(x) = 1 - (x-3) + (x-3)^2 - (x-3)^3 + \cdots + (-1)^n(x-3)^n + \cdots$$

(b) Integrate term by term.
$$\ln|x-2| = \int_3^x \frac{1}{t-2}\,dt$$
$$= \left[t - \frac{1}{2}(t-3)^2 + \frac{1}{3}(t-3)^3 - \frac{1}{4}(t-3)^4 + \cdots + (-1)^n\frac{(t-3)^{n+1}}{n+1} + \cdots\right]_3^x$$
$$= (x-3) - \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} - \frac{(x-3)^4}{4} + \cdots + (-1)^n\frac{(x-3)^{n+1}}{n+1} + \cdots$$

(c) Evaluate at $x = 3.5$. This is the alternating series
$$\frac{1}{2} - \frac{1}{2^2\cdot 2} + \frac{1}{2^3\cdot 3} - \cdots + (-1)^n\frac{1}{2^{n+1}(n+1)} + \cdots$$
By the Alternating Series Estimation Theorem, since the size of the third term is $\frac{1}{24} < 0.05$, the first two terms will suffice. The estimate for $\ln\left(\frac{3}{2}\right)$ is 0.375.

61. (a) Substitute $-2x^2$ for x in the Maclaurin series for e^x given at the end of Section 9.2.
$$e^{-2x^2} = 1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \cdots + \frac{(-2x^2)^n}{n!} + \cdots$$
$$= 1 - 2x^2 + 2x^4 - \frac{4x^6}{3} + \cdots + (-1)^n\frac{2^n x^{2n}}{n!} + \cdots$$

(b) Use the Ratio Test:
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}x^{2n+2}}{(n+1)!}\cdot\frac{n!}{2^n x^{2n}}$$
$$= \lim_{n\to\infty}\frac{2x^2}{n+1} = 0$$
The series converges for all real numbers, so the interval of convergence is $(-\infty, \infty)$.

(c) This is an alternating series. The difference will be bounded by the magnitude of the fifth term, which is $\frac{(2x^2)^4}{4!} = \frac{2x^8}{3}$. Since $-0.6 \le x \le 0.6$, this term is less than $\frac{2(0.6)^8}{3}$ which is less than 0.02.

62. (a)
$$f(x) = x^2\left(\frac{1}{1+x}\right)$$
$$= x^2(1 - x + x^2 + \cdots + (-x)^n + \cdots)$$
$$= x^2 - x^3 + x^4 + \cdots + (-1)^n x^{n+2} + \cdots$$

(b) No. At $x = 1$, the series is $\sum_{n=0}^{\infty}(-1)^n$ and the partial sums form the sequence 1, 0, 1, 0, 1, 0, ..., which has no limit.