Tangents, Arcs, and Chords

Objectives

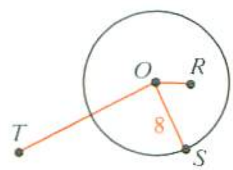
1. Define a circle, a sphere, and terms related to them.
2. Recognize circumscribed and inscribed polygons and circles.
3. Apply theorems that relate tangents and radii.
4. Define and apply properties of arcs and central angles.
5. Apply theorems about the chords of a circle.

7-1 Basic Terms

A **circle** ($\odot$) is the set of points in a plane that are a given distance from a given point in the plane. The given point is the **center**, and the given distance is the **radius**. A segment that joins the center to a point on the circle is called *a radius*. All radii of a circle are congruent. The circle shown, with center O, is called circle O ($\odot O$).

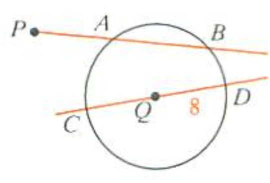
The radius of $\odot O$ is 8.
$\overline{OS}$ is a radius of $\odot O$.

R lies inside the circle.	$OR < 8$
S lies on the circle.	$OS = 8$
T lies outside the circle.	$OT > 8$



A **chord** is a segment that joins two points on a circle. A **diameter** is a chord that passes through the center. A **secant** is a line that contains a chord.

Chords: $\overline{AB}$, $\overline{CD}$
Diameter: $\overline{CD}$
Secants: $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$



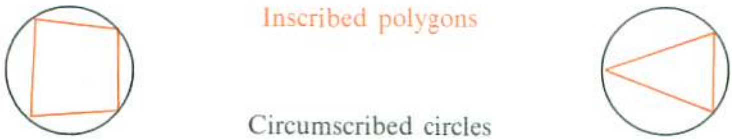
A ray or segment containing a chord is often also called a secant. For example, you can refer to $\overline{PB}$ in the diagram as a secant drawn to $\odot Q$ from external point P.

Like the word *radius*, the word *diameter* can refer to the length of a segment as well as to the segment itself. From the definitions it follows that the diameter of a circle equals twice the radius. Thus $\overline{CD}$ is *a* diameter of $\odot Q$ and *the* diameter of $\odot Q$ is 16.

Congruent circles are circles that have congruent radii. By definition, radii of congruent circles are congruent.

Concentric circles are circles that lie in the same plane and have the same center. The rings of the target illustrate concentric circles.

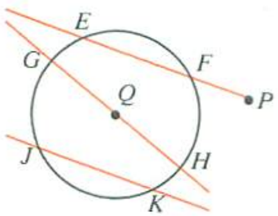
A polygon is **inscribed in a circle** and the circle is **circumscribed about the polygon** when each vertex of the polygon lies on the circle.



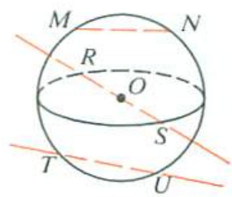
Remove the phrase *in a plane* from the definition of a circle and you have the definition of a *sphere*. A **sphere** is the set of points that are a given distance from a given point. Many of the terms used for circles are also used for spheres.

Classroom Exercises

1. Name two radii, a diameter, three chords, and two secants of $\odot Q$.



2. Name two radii, a diameter, three chords, and two secants of sphere O.



3. See the figure for Exercise 1. Name a secant to $\odot Q$ from external point P.

4. Write a definition of concentric spheres.

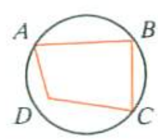
5. What is the diameter of a circle with radius 8? 5.2? $4\sqrt{3}$? j?

6. What is the radius of a sphere with diameter 14? 13? 5.6? $6n$?

7. Draw a circle and several parallel chords. What do you think is true of the midpoints of all such chords?

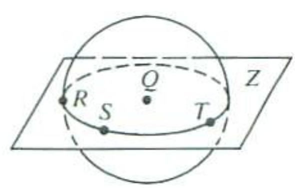
8. Point Z lies on $\odot X$. How many chords can you draw that contain Z? How many diameters? How many radii?

9. Explain why it is not correct to call $ABCD$ an inscribed quadrilateral.



10. Plane Z passes through the center of sphere Q.

- **a.** Explain why $QR = QS = QT$.
- **b.** Explain why the intersection of the plane and the sphere is a circle. (The intersection of a sphere with any plane passing through the center of the sphere is called a **great circle** of the sphere.)
- **c.** On a globe, which of the following are great circles: (1) the equator, (2) the Arctic Circle, and (3) the circle formed by the 0° meridian (also called the prime meridian) and the 180° meridian?



Written Exercises

Point W lies outside $\odot O$ and point X lies inside. In how many points does the figure named intersect the circle?

A 1. $\overline{WX}$ 2. $\overrightarrow{WX}$ 3. $\overrightarrow{XW}$ 4. $\overline{OX}$

Point Y lies outside sphere Q and point Z lies inside. In how many points does the figure named intersect the sphere?

5. $\overline{YZ}$ 6. $\overrightarrow{YQ}$ 7. $\overrightarrow{ZY}$ 8. $\overline{QZ}$

The radius of sphere P is 12. Plane M cuts the sphere in a circle. Tell what you can about the center and radius of the circle if:

9. Plane M passes through P.
10. Plane M does not pass through P.

For each exercise draw a circle and inscribe the figure named in the circle. If a polygon of the type named can't be inscribed, write *not possible*.

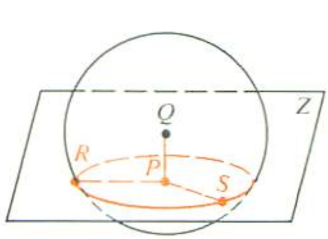
11. A rectangle
12. A trapezoid
13. An obtuse triangle
14. A nonrectangular parallelogram
15. An acute isosceles triangle
16. A quadrilateral $PQRS$, with $\overline{PR}$ a diameter

For each exercise draw $\odot O$ with radius 12. Then draw radii $\overline{OA}$ and $\overline{OB}$ to form an angle with the measure named. Find the length of $\overline{AB}$.

B 17. $m\angle AOB = 90$ 18. $m\angle AOB = 180$ 19. $m\angle AOB = 60$ 20. $m\angle AOB = 120$

21. Write a definition of radius of a sphere.
22. Write a definition of congruent spheres.
23. A plane 6 units from the center of a sphere of radius 10 intersects the sphere in a circle. Find the radius of the circle.
24. A plane 15 cm from the center of a sphere cuts the sphere in a circle with a radius of 8 cm. Find the radius of the sphere.

C 25. Two spheres with radii of 6 cm and 4 cm have centers that are 8 cm apart. Find the radius of the circle in which the spheres intersect.
26. Prove: A line intersects a circle in at most two points. (*Hint:* Write an indirect proof.)
27. Exercises 23 and 24 assume the following theorem: If a plane intersects a sphere in more than one point, the intersection is a circle. Prove this theorem. (*Hint:* The case where the plane, Z, passes through the center of the sphere, Q, is covered in Classroom Exercise 10. If Z does not pass through Q, draw a perpendicular from Q to Z, intersecting Z in point P. (You may assume that this is possible.) Let R and S be any two points on the intersection of the sphere and the plane. Show that $\overline{PR} \cong \overline{PS}$.)



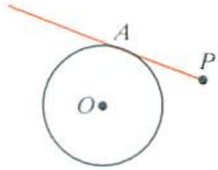
7-2 Tangents

A **tangent** to a circle is a line in the plane of the circle that meets the circle in exactly one point, called the **point of tangency.**

$\overleftrightarrow{AP}$ is tangent to $\odot O$.

$\odot O$ is tangent to $\overleftrightarrow{AP}$.

A is the point of tangency.



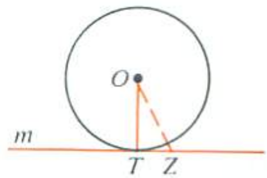
The tangent ray $\overrightarrow{PA}$ and the tangent segment $\overline{PA}$ are often called simply tangents. $\overline{PA}$ is a tangent (or tangent segment) to $\odot O$ from external point P.

Theorem 7-1

If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.

Given: Line m is tangent to $\odot O$ at point T.

Prove: $\overline{OT} \perp m$



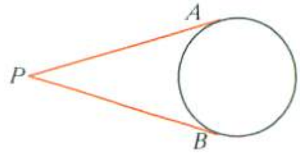
Indirect proof:

Assume temporarily that $\overline{OT}$ is not perpendicular to m. Draw the line through O that is perpendicular to m and call it $\overleftrightarrow{OZ}$. Because the perpendicular segment from O to m is the shortest segment from O to m, we have $OZ < OT$. Since m intersects $\odot O$ only in point T, Z must lie outside $\odot O$, and therefore $OZ > OT$. The statements $OZ < OT$ and $OZ > OT$ are contradictory. This shows that what we temporarily assumed, that $\overline{OT}$ is not perpendicular to m, must be false. We conclude that $\overline{OT} \perp m$.

Corollary

Tangents to a circle from a point are congruent.

In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to the circle at A and B. The corollary tells us that $\overline{PA} \cong \overline{PB}$. For a proof, see Classroom Exercise 7.



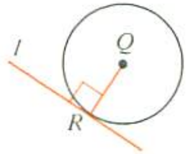
The proof of the following converse to Theorem 7-1 is left as Exercise 23.

Theorem 7-2

If a line in the plane of a circle is perpendicular to a radius at its outer endpoint, then the line is tangent to the circle.

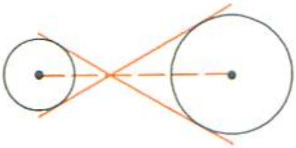
Given: Line l in the plane of $\odot Q$;
$l \perp$ radius $\overline{QR}$ at R

Prove: l is tangent to $\odot Q$.

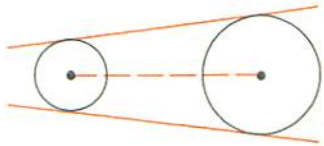


A line that is tangent to each of two coplanar circles is called a **common tangent.**

Common *internal* tangents intersect the segment joining the centers.

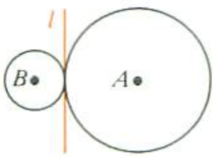


Common *external* tangents do *not* intersect the segment joining the centers.

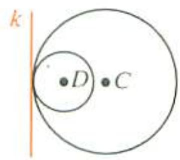


Two *circles are tangent* to each other when they are coplanar and are tangent to the same line at the same point.

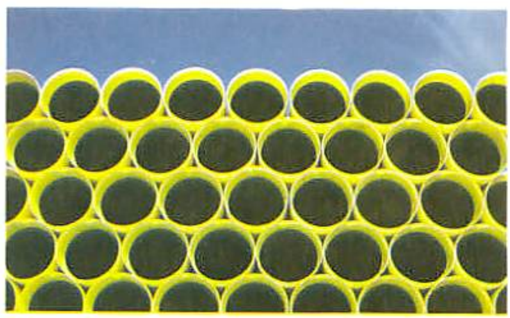
$\odot A$ and $\odot B$ are *externally* tangent.



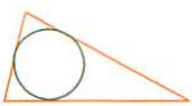
$\odot C$ and $\odot D$ are *internally* tangent.



The ends of the plastic industrial pipes shown in the photograph illustrate externally tangent circles. Notice that when a circle is surrounded by tangent circles of the same radius, six of these circles fit exactly around the inner circle.

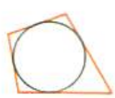


When each side of a polygon is tangent to a circle, the polygon is said to be **circumscribed about the circle.** The circle is **inscribed in the polygon.**



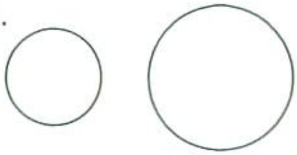
Circumscribed polygons

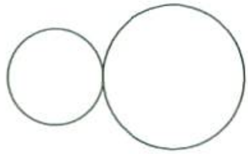
Inscribed circles

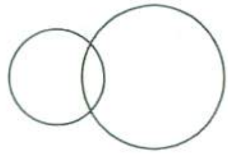


Classroom Exercises

How many common external tangents and how many common internal tangents can be drawn to the two circles?

1. 

2. 

3. 

4. 

5. 

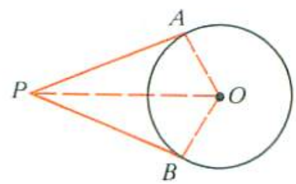
6. 

7. Write a proof of the corollary to Theorem 7-1.

Given: $\overline{PA}$ and $\overline{PB}$ are tangents to $\odot O$.

Prove: $\overline{PA} \cong \overline{PB}$

Plan for Proof: Draw $\overline{AO}$, $\overline{BO}$, and $\overline{PO}$. Show that two triangles are congruent and use corresponding parts.



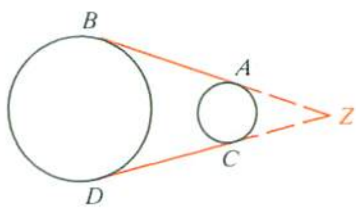
8. State and prove a relation between $\angle APB$ and $\angle AOB$.

Written Exercises

A **1.** Copy and complete this proof that the common external tangent segments to two noncongruent circles are congruent.

Given: Tangents $\overline{AB}$ and $\overline{CD}$

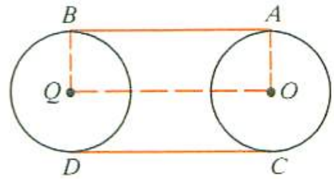
Prove: $\overline{AB} \cong \overline{CD}$



Proof:

Statements	Reasons
1. Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, intersecting at Z.	1. Through any two points there is _?_.
2. $ZA + AB = ZB$; $ZC + CD = ZD$	2. _?_
3. $ZB = ZD$	3. _?_
4. $ZA + AB = ZC + CD$	4. _?_
5. $ZA = ZC$	5. _?_
6. $AB = CD$, or $\overline{AB} \cong \overline{CD}$	6. _?_

2. Suppose, in Exercise 1, that the circles are congruent. Then the tangent lines won't meet. Supply reasons for these key steps of the proof that the common tangent segments are congruent for this special case.



Given: Tangents $\overline{AB}$ and $\overline{CD}$;
$\odot O \cong \odot Q$

Prove: $\overline{AB} \cong \overline{CD}$

1. $\overline{OA} \perp \overline{AB}$ and $\overline{QB} \perp \overline{AB}$
2. $\overline{OA} \parallel \overline{QB}$
3. $\overline{OA} \cong \overline{QB}$
4. Quad. $AOQB$ is a ▱.
5. $\overline{AB} \cong \overline{OQ}$

By a similar proof, $\overline{OQ} \cong \overline{CD}$.

6. $\overline{AB} \cong \overline{CD}$

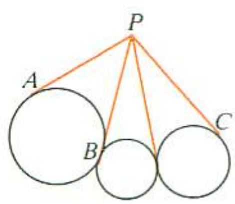
Draw two circles, with all their common tangents, so that the number of common tangents is the stated number.

3. one **4.** two **5.** three **6.** four

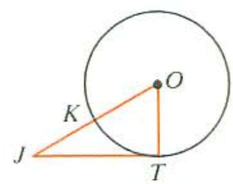
7. How many circles can be tangent to a given line at a given point on the line?

8. Circles O and Q are tangent at point Z. Suppose you draw a different circle tangent to $\odot O$ at Z. What can you say about the new circle and $\odot Q$?

9. The diagram shows tangent circles and lines.
$PA = 10$ $PB = \underline{\ ?\ }$ $PC = \underline{\ ?\ }$



Ex. 9



Exs. 10–12

In the diagram for Exercises 10–12, $\overline{JT}$ is tangent to $\odot O$.

10. If $JO = 13$ and $OT = 5$, then $JT = \underline{\ ?\ }$.

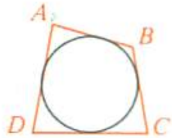
11. If $m\angle OJT = 30$ and $JO = 20$, then $JT = \underline{\ ?\ }$.

12. If $JK = 9$ and $KO = 8$, then $JT = \underline{\ ?\ }$.

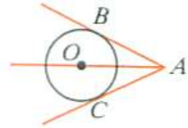
B **13.** Discover and prove a theorem about two lines tangent to a circle at the endpoints of a diameter.

14. State, without proof, a theorem about spheres related to the theorem in Exercise 13.

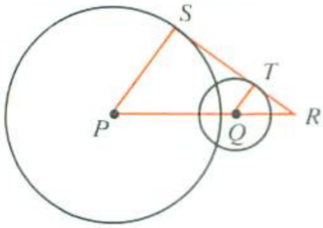
15. Quad. $ABCD$ is circumscribed about a circle. Discover and prove a relationship between $AB + DC$ and $AD + BC$.



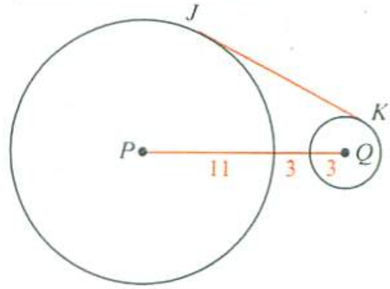
16. Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are tangent to $\odot O$. Discover and prove a theorem about $\overrightarrow{AO}$ and $\angle BAC$.



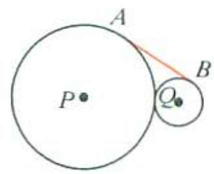
17. $\overline{SR}$ is tangent to $\odot P$ and $\odot Q$.
$QT = 6$; $TR = 8$; $PR = 30$
$PS =$ __?__ $PQ =$ __?__ $ST =$ __?__



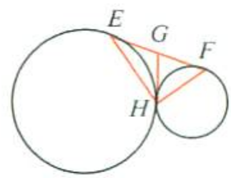
18. $\overline{JK}$ is tangent to $\odot P$ and $\odot Q$.
$JK =$ __?__ (*Hint:* What kind of quadrilateral is $JPQK$?)



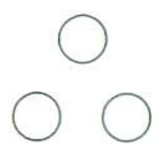
19. Circles P and Q have radii 6 and 2 and are tangent to each other. Find the length of their common external tangent $\overline{AB}$. (*Hint:* Draw $\overline{PQ}$, $\overline{PA}$, and $\overline{QB}$.)



20. Given: Two tangent circles; $\overline{EF}$ is a common external tangent; $\overline{GH}$ is the common internal tangent.
Prove: $\angle EHF$ is a rt. $\angle$.



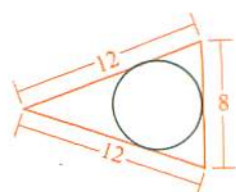
21. Three circles are shown. How many circles tangent to all three of the given circles can be drawn?



C 22. Suppose the three circles represent three spheres.
 a. How many planes tangent to each of the spheres can be drawn?
 b. How many spheres tangent to each of the three spheres can be drawn?

23. Prove Theorem 7-2. (*Hint:* Write an indirect proof.)

24. Find the radius of the circle inscribed in the triangle.

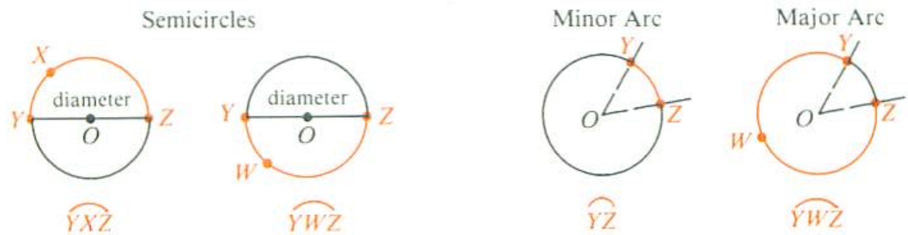


25. A circle inscribed in right $\triangle ABC$ is tangent to hypotenuse $\overline{AB}$ at K. $AK = 20$ and $BK = 6$. Find the sides of $\triangle ABC$.

26. The diameter of the circle inscribed in a certain right triangle is 6 while the diameter of the circumscribed circle is 17. Find the sides of the triangle.

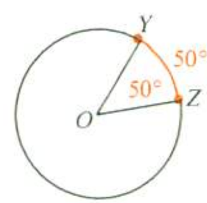
7-3 Arcs and Central Angles

An *arc* is an unbroken part of a circle. Two points Y and Z on a circle O are always the endpoints of two arcs. If $\overline{YZ}$ is a diameter, the two arcs are called **semicircles.** Otherwise, Y and Z and the points of $\odot O$ in the interior of $\angle YOZ$ form a **minor arc.** Y and Z and the remaining points of $\odot O$ form a **major arc.**



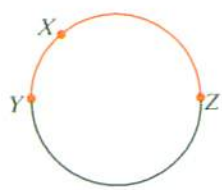
A minor arc is named by its endpoints: $\widehat{YZ}$ is read "arc YZ." You use three letters to name a semicircle or a major arc: $\widehat{YWZ}$ is read "arc YWZ."

A **central angle** of a circle is an angle with its vertex at the center of the circle. A central angle that intersects a minor arc at its endpoints is called the central angle of that arc. Angle YOZ is the central angle of minor arc YZ shown above.



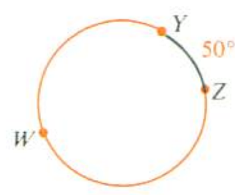
The **measure of a minor arc** is defined to be the measure of its central angle. To indicate that the measure of $\widehat{YZ}$ equals the measure of central angle YOZ you can write $m\widehat{YZ} = m\angle YOZ$. For example, if $m\angle YOZ = 50$, then $m\widehat{YZ} = 50$.

The **measure of a semicircle** is 180. The **measure of a major arc** will always be greater than 180. It is found as shown below.



Semicircle

$$m\widehat{YXZ} = 180$$



Major Arc

$$m\widehat{YWZ} = 360 - m\widehat{YZ} = 360 - 50 = 310$$

Adjacent nonoverlapping arcs of a circle are arcs that have exactly one point in common. The following postulate is used when we compute arc measures by adding the measures of adjacent arcs.

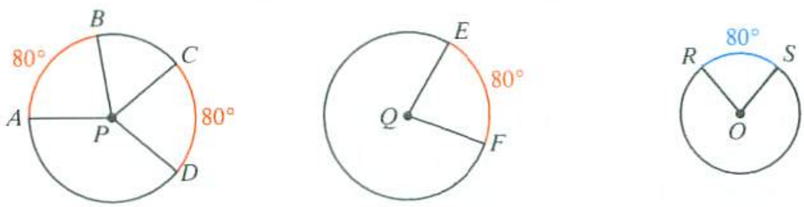
Postulate 16 Arc Addition Postulate

The measure of the arc formed by two adjacent nonoverlapping arcs is the sum of the measures of these two arcs.

For example, applying Postulate 16 to the circle shown at the right, we have

$$m\overset{\frown}{AD} + m\overset{\frown}{DC} = m\overset{\frown}{AC}$$
$$m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{ABC}$$
$$m\overset{\frown}{ABC} + m\overset{\frown}{CD} = m\overset{\frown}{ABD}$$

Congruent arcs are arcs, in the same circle or in congruent circles, that have equal measures. In the diagram below, $\odot P$ and $\odot Q$ are congruent circles and $\overset{\frown}{AB} \cong \overset{\frown}{CD} \cong \overset{\frown}{EF}$. However, $\overset{\frown}{EF}$ is not congruent to $\overset{\frown}{RS}$ even though both arcs have the same degree measure, because $\odot Q$ is not congruent to $\odot O$.



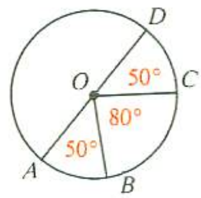
Notice that each of the congruent arcs above has an 80° central angle, so these congruent arcs have congruent central angles. And you can see in the photograph that the congruent central angles formed by adjacent spokes cut off congruent arcs along the rim of the wheel.

This relationship between congruence of minor arcs and congruence of their central angles is stated by Theorem 7-3 on the next page. The theorem will be proved in Classroom Exercises 14 and 15.

Theorem 7-3

In the same circle or in congruent circles, two minor arcs are congruent if and only if their central angles are congruent.

Classroom Exercises



1. Using the letters shown in the diagram, name:

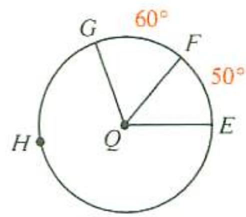
a. Two central angles **b.** A semicircle

c. Two minor arcs **d.** Two major arcs

In Exercises 2–7 find the measure of the arc.

2. $\widehat{AB}$ **3.** $\widehat{AC}$ **4.** $\widehat{ABD}$

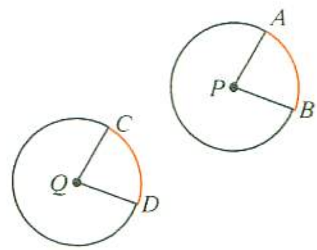
5. $\widehat{BAD}$ **6.** $\widehat{CDA}$ **7.** $\widehat{CDB}$



In Exercises 8–13 find the measure of the angle or the arc named.

8. $\angle GQF$ **9.** $\angle EQF$ **10.** $\angle GQE$

11. $\widehat{GE}$ **12.** $\widehat{GHE}$ **13.** $\widehat{EHF}$



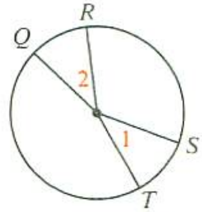
14. Complete the proof of the "only if" part of Theorem 7-3 for the case of two congruent circles. That is, prove that in congruent circles two minor arcs are congruent only if their central angles are congruent. Give reasons for each statement.

Given: $\odot P \cong \odot Q$; $\widehat{AB} \cong \widehat{CD}$

Prove: $\angle APB \cong \angle CQD$

Proof:

Statements	Reasons
1. $m\widehat{AB} = m\angle APB$; $m\widehat{CD} = m\angle CQD$	1. __?__
2. $\widehat{AB} \cong \widehat{CD}$, or $m\widehat{AB} = m\widehat{CD}$	2. __?__
3. $m\angle APB = m\angle CQD$, or $\angle APB \cong \angle CQD$	3. __?__



15. Write a proof for the "if" part of Theorem 7-3 for congruent angles in the same circle. That is, prove that two minor arcs of a circle are congruent if their central angles are congruent.

Given: $\angle 1 \cong \angle 2$

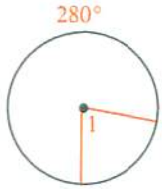
Prove: $\widehat{TS} \cong \widehat{QR}$

Written Exercises

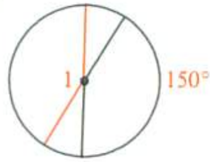
Find the measure of central $\angle 1$.

A 1.

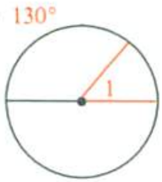
2.



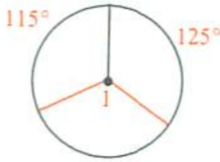
3.



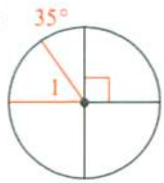
4.



5.



6.

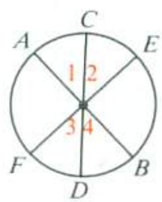


7. At 11 o'clock the hands of a clock form an angle of __?__ °.

8. The hands of a clock form a 120° angle at __?__ o'clock and at __?__ o'clock.

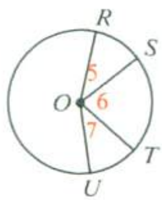
9. **a.** Draw a circle. Place points A, B, and C on it in such positions that $m\widehat{AB} + m\widehat{BC}$ does not equal $m\widehat{AC}$.
   b. Does your example in part (a) contradict Postulate 16?

In Exercises 10 and 11, $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ are diameters.

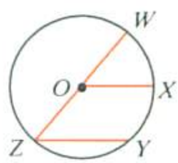


10. Given: $\widehat{AC} \cong \widehat{CE}$
    Prove: $\angle 3 \cong \angle 4$

11. Given: $\angle 1 \cong \angle 2$
    Prove: $\widehat{BD} \cong \widehat{DF}$



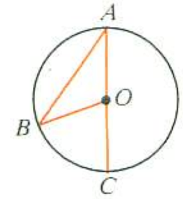
12. Given: $\odot O$ with $m\angle 5 = m\angle 7$
    Prove: $\widehat{RT} \cong \widehat{SU}$

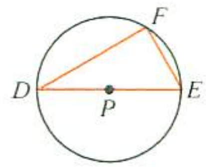


B 13. Given: $\overline{WZ}$ is a diameter of $\odot O$;
    $\overline{OX} \parallel \overline{ZY}$
    Prove: $\widehat{WX} \cong \widehat{XY}$
    (*Hint:* Draw $\overline{OY}$.)

14. Given: $\overline{WZ}$ is a diameter of $\odot O$;
    $m\widehat{WX} = m\widehat{XY} = n$
    Prove: $m\angle Z = n$

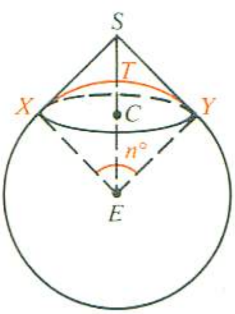
15. $\overline{AC}$ is a diameter of $\odot O$.
   a. If $m\angle A = 35$, then $m\angle B = \underline{\ ?\ }$, $m\angle BOC = \underline{\ ?\ }$, and $m\widehat{BC} = \underline{\ ?\ }$.
   b. If $m\angle A = n$, then $m\widehat{BC} = \underline{\ ?\ }$.
   c. If $m\widehat{BC} = 6k$, then $m\angle A = \underline{\ ?\ }$.





16. $\overline{DE}$ is a diameter of $\odot P$ and $m\widehat{EF} = n$.
$m\angle DEF = \underline{\ ?\ }$.
(*Hint:* Draw $\overline{PF}$.)

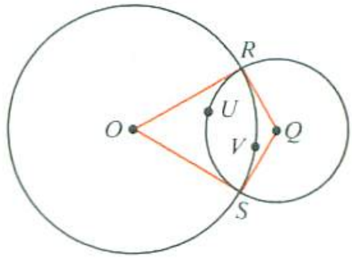
The diagram, not drawn to scale, shows satellite S above the Earth, represented as sphere E. All lines tangent to the Earth from S touch the Earth at points on a circle with center C. Any two points on the Earth's surface on or above that circle can communicate with each other via S. X and Y are as far apart as two communication points can be. The Earth distance between X and Y equals the length of $\widehat{XTY}$, which equals $\frac{n}{360} \cdot$ circumference of the Earth. That circumference is approximately 40,200 km and the radius of the Earth is approximately 6400 km.



C 17. The photograph above shows the view from Gemini V looking north over the Gulf of California toward Los Angeles. The orbit of Gemini V ranged from 160 km to 300 km above the Earth. Take S to be 300 km above the Earth. That is, $ST = 300$ km. Find the Earth distance, rounded to the nearest 100 km, between X and Y. (*Hint:* Since you can find the value of $\cos \frac{n^\circ}{2}$ you can determine n°.)

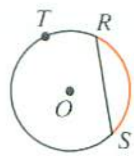
18. Repeat Exercise 17, but with S twice as far from the Earth. Note that the distance between X and Y is not twice as great as before.

19. Given: $\odot O$ and $\odot Q$ intersect at R and S;
$m\widehat{RVS} = 60$; $m\widehat{RUS} = 120$
Prove: $\overline{OR}$ is tangent to $\odot Q$;
$\overline{QR}$ is tangent to $\odot O$.



20. Given: $\overline{AB}$ is a diameter of $\odot Z$; points J and K lie on $\odot Z$ with $m\widehat{AJ} = m\widehat{BK}$. Discover and prove something about $\overline{JK}$. (*Hint:* There are two possibilities, depending on whether $\widehat{AJ}$ and $\widehat{BK}$ lie on the same side of $\overline{AB}$ or on opposite sides. So your statement will be of the *either . . . or* type.)

7-4 Arcs and Chords



In $\odot O$, $\overline{RS}$ cuts off two arcs, $\widehat{RS}$ and $\widehat{RTS}$. We speak of $\widehat{RS}$, the minor arc, as being *the arc of chord RS*.

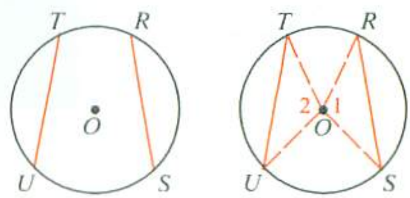
Theorem 7-4

In the same circle or in congruent circles:

(1) Congruent chords have congruent arcs.

(2) Congruent arcs have congruent chords.

We outline the proof of part (1) for one circle.



Given: $\odot O$; $\overline{RS} \cong \overline{TU}$

Prove: $\widehat{RS} \cong \widehat{TU}$

Outline of proof:

1. Draw radii $\overline{OR}$, $\overline{OS}$, $\overline{OT}$, and $\overline{OU}$.
2. $\overline{OR} \cong \overline{OT}$; $\overline{OS} \cong \overline{OU}$ (Why?)
3. $\overline{RS} \cong \overline{TU}$ (Given)
4. $\triangle ROS \cong \triangle TOU$ (Why?)
5. $\angle 1 \cong \angle 2$ (Corres. __?__.)
6. $\widehat{RS} \cong \widehat{TU}$ (Why?)

The next theorem involves the idea of bisecting an arc. Y is called the midpoint of $\widehat{XYZ}$ if $\widehat{XY} \cong \widehat{YZ}$. Any line, segment, or ray that contains Y bisects $\widehat{XYZ}$.

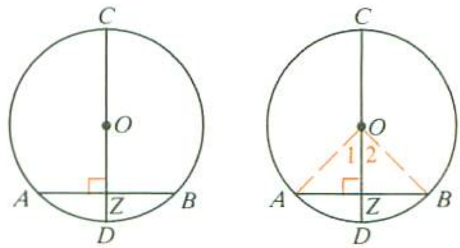
Theorem 7-5

A diameter that is perpendicular to a chord bisects the chord and its arc.

Given: $\odot O$; $\overline{CD} \perp \overline{AB}$

Prove: $\overline{AZ} \cong \overline{BZ}$; $\widehat{AD} \cong \widehat{BD}$

Plan for Proof: Draw $\overline{OA}$ and $\overline{OB}$. Use the HL Theorem to prove that right triangles OZA and OZB are congruent. Then use corresponding parts of congruent triangles to show that $\overline{AZ} \cong \overline{BZ}$ and $\angle 1 \cong \angle 2$. Finally, apply the theorem that congruent central angles have congruent arcs.



You will prove part (1) of the next theorem as Classroom Exercise 2.

Theorem 7-6

In the same circle or in congruent circles:

(1) Congruent chords are equally distant from the center (or centers).

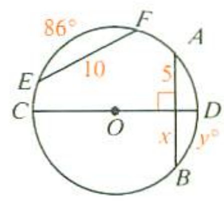
(2) Chords equally distant from the center (or centers) are congruent.

Example 1 Find the values of x and y.

Solution Diameter $\overline{CD}$ bisects chord $\overline{AB}$, so $x = 5$. (Theorem 7-5)

$\overline{AB} \cong \overline{EF}$, so $m\widehat{AB} = 86$. (Theorem 7-4)

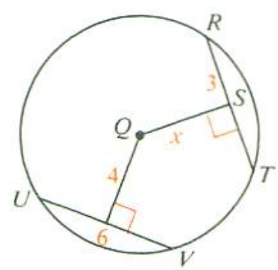
Diameter $\overline{CD}$ bisects $\widehat{AB}$, so $y = 43$. (Theorem 7-5)



Example 2 Find the value of x.

Solution S is the midpoint of $\overline{RT}$, so $RT = 6$. (Theorem 7-5)

$\overline{RT} \cong \overline{UV}$, so $x = 4$. (Theorem 7-6)



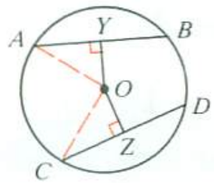
Classroom Exercises

1. See Theorem 7-4 on page 306. Which statements in the outline of the proof would you modify to get a proof of part (1) for congruent circles?

2. Supply reasons to complete a proof of Theorem 7-6, part (1), for one circle.

Given: $\odot O$; $\overline{AB} \cong \overline{CD}$;
$\overline{OY} \perp \overline{AB}$; $\overline{OZ} \perp \overline{CD}$

Prove: $OY = OZ$



Proof:

Statements	Reasons
1. Draw radii $\overline{OA}$ and $\overline{OC}$.	1. ?
2. $\overline{AB} \cong \overline{CD}$, or $AB = CD$	2. ?
3. $\frac{1}{2}AB = \frac{1}{2}CD$	3. ?
4. $AY = \frac{1}{2}AB$; $CZ = \frac{1}{2}CD$	4. ?
5. $AY = CZ$	5. ?
6. $OA = OC$	6. ?
7. rt. $\triangle OYA \cong$ rt. $\triangle OZC$	7. ?
8. $OY = OZ$	8. ?

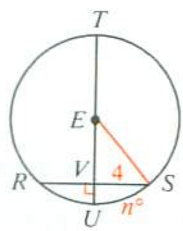
3. In $\odot E$, diameter $\overline{TU} \perp \overline{RS}$; $VS = 4$; $m\widehat{US} = n$.

a. $RV =$? and $m\widehat{RU} =$?.

b. In terms of n, $m\widehat{ST} =$? and $m\widehat{RT} =$?.

c. From part (b) we conclude that $\widehat{RT} \cong$?.

d. Suppose $TU = 10$. Then $EV =$?.



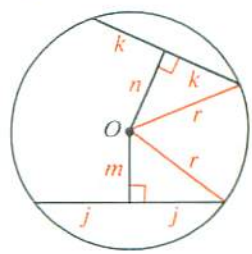
4. In $\odot O$, the chords have unequal lengths, with $2j > 2k$.

a. From $m^2 + j^2 = r^2$ we get $m = \sqrt{? - ?}$.

b. We also have $n = \sqrt{? - ?}$.

c. Because $j > k$, $\sqrt{r^2 - j^2}$ $\underset{(<, =, >)}{?}$ $\sqrt{r^2 - k^2}$.

d. By substitution in (c) we get m $\underset{(<, =, >)}{?}$ n.

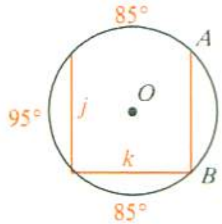


5. Exercise 4 provides a condensed proof of a theorem not stated in this textbook. Complete the statement: If two chords of a circle have unequal lengths, then the longer chord is ?.

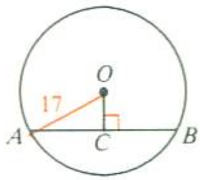
Written Exercises

In the diagrams that follow, O is the center of each circle.

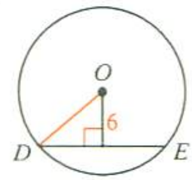
A

1. 

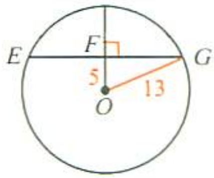
$AB = \underline{\ ?\ }$

2. 

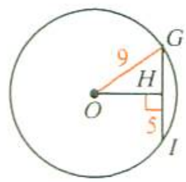
$AB = 30$; $OC = \underline{\ ?\ }$

3. 

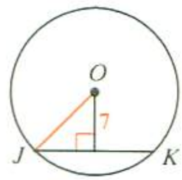
$DE = 16$; $OD = \underline{\ ?\ }$

4. 

$EG = \underline{\ ?\ }$

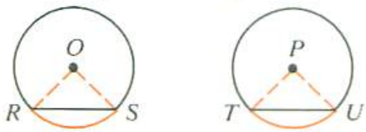
5. 

$OH = \underline{\ ?\ }$

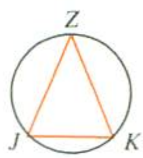
6. 

$JK = 14$; $OJ = \underline{\ ?\ }$

7. Prove part 2 of Theorem 7–4 for congruent circles. First list what is given and what you are to prove.

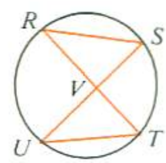


8. **a.** Given: $\widehat{JZ} \cong \widehat{KZ}$
Prove: $\angle J \cong \angle K$
b. Is the converse of part (a) true?



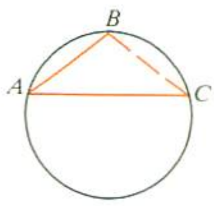
B

9. Given: $\overline{RS} \cong \overline{UT}$
Prove: $\overline{RT} \cong \overline{US}$
(*Hint:* Apply Theorem 7–4 and the Arc Addition Postulate.)

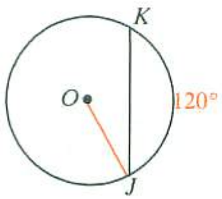


10. Given: $\widehat{RS} \cong \widehat{UT}$; $\angle R \cong \angle U$
Prove: $\overline{VS} \cong \overline{VT}$ and $\overline{RV} \cong \overline{UV}$

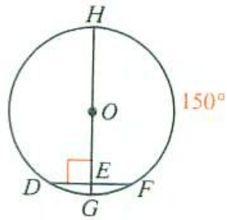
11. The informal statement "When you double the length of an arc you double the length of the chord" may seem at first glance to be true. But use the figure, in which $m\widehat{AC} = 2 \cdot m\widehat{AB}$, to show that $AC \neq 2 \cdot AB$.



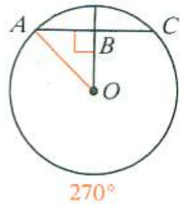
12. **a.** Draw three generous-sized circles and inscribe a different-shaped quadrilateral $ABCD$ in each.
b. Use a protractor to measure all the angles.
c. Compare $\angle A$ and $\angle C$, $\angle B$ and $\angle D$.
d. Although you haven't proved anything in this exercise, you should wonder about a possible theorem. State the theorem.

13. 

If $OJ = 12$, $JK =$ __?__.

14. 

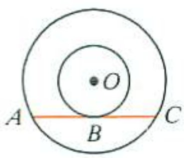
If $OE = 8\sqrt{3}$, $HG =$ __?__.

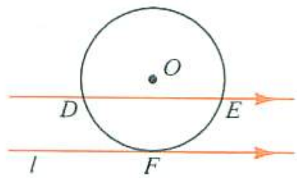
15. 

If $OA = 9$, $BC =$ __?__.

16. The radius of a sphere is j. The distance from the center of the sphere to a certain chord is k. How long is the chord? Answer in terms of j and k.

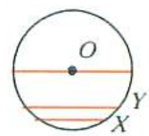
State and prove a theorem suggested by the figure.

C 17. 

18. 

19. Investigate the possibility, given a circle, of drawing two chords whose lengths are in the ratio 1:2 and whose distances from the center are in the ratio 2:1. If the chords can be drawn, find the length of each in terms of the radius. If not, prove that the figure is impossible.

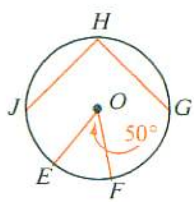
20. Three parallel chords of $\odot O$ are drawn as shown. Their lengths are 20, 16, and 12 cm. Find, to the nearest tenth of a centimeter, the length of chord $\overline{XY}$ (not shown).



Self-Test 1

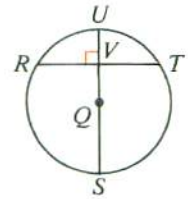
1. Sketch a triangle inscribed in one circle and sketch a quadrilateral circumscribed about another circle.
2. Circles O and Q are congruent circles. The radius of $\odot O$ is 8. The diameter of $\odot Q$ is __?__.
3. Two circles intersect in two points. How many common tangents can be drawn to the circles?
4. A plane passes through the common center of two concentric spheres. Describe the intersection of the plane and the two spheres.

Points E, F, G, H, and J lie on $\odot O$.



5. $m\widehat{EF} =$ __?__
6. $m\widehat{EHF} =$ __?__
7. Suppose $\overline{JH} \cong \overline{HG}$. State the theorem that supports the conclusion that $\widehat{JH} \cong \widehat{HG}$.

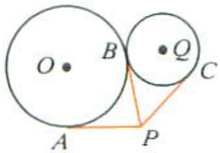
In $\odot Q$, diameter $\overline{US} \perp \overline{RT}$.



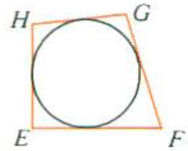
8. If $m\widehat{RST} = 220$, $m\widehat{UT} =$ __?__.

9. If $RT = 16$ and $QS = 10$, then $QV =$ __?__.

10. Given: Tangent circles O and Q with tangents $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ as shown.
Prove: $\overline{PA} \cong \overline{PC}$



Ex. 10



Ex. 11

11. In the circumscribed quadrilateral, $EF = 16$, $FG = 15$, and $GH = 12$. $HE =$ __?__.

Angles and Segments

Objectives

1. Solve problems and prove statements involving inscribed angles.
2. Solve problems and prove statements involving angles formed by chords, secants, and tangents.
3. Solve problems involving lengths of chords, secant segments, and tangent segments.

7-5 Inscribed Angles

In the figures below, we say that the angles *intercept* the arcs shown in color.

An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle. In the diagrams above, only $\angle 1$ and $\angle 2$ are inscribed angles. $\angle 1$ intercepts a minor arc, and $\angle 2$ intercepts a major arc. Some inscribed angles intercept semicircles. $\angle 3$ intercepts two arcs and $\angle 4$ intercepts one arc.

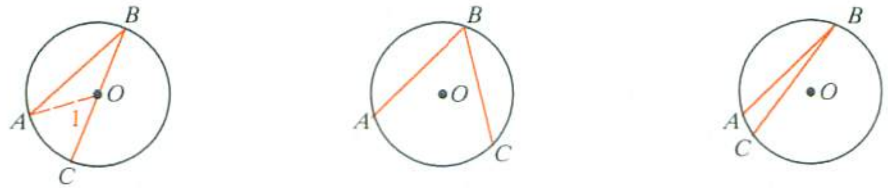
The next theorem compares the measure of an inscribed angle with the measure of its intercepted arc.

Theorem 7-7

The measure of an inscribed angle is equal to half the measure of its intercepted arc.

Given: $\angle ABC$ inscribed in $\odot O$

Prove: $m\angle ABC = \frac{1}{2}m\widehat{AC}$



Case I: Point O lies on $\angle ABC$.

Case II: Point O lies inside $\angle ABC$.

Case III: Point O lies outside $\angle ABC$.

Outline of Proof of Case I:

1. Draw radius $\overline{OA}$. (Through any two points there is exactly one line.)
2. $\overline{OB} \cong \overline{OA}$ (All radii of a circle are congruent.)
3. $m\angle A = m\angle B$ (If two sides of a $\triangle$ are $\cong$, __?__.)
4. $m\angle A + m\angle B = m\angle 1$ (The measure of an exterior angle of a $\triangle$ = __?__.)
5. $m\angle B + m\angle B = m\angle 1$ (Why?)
6. $m\angle B = \frac{1}{2}m\angle 1$ (Division Property of =)
7. $m\widehat{AC} = m\angle 1$ (Why?)
8. $m\angle B = \frac{1}{2}m\widehat{AC}$ (Substitution Property)

The proofs of Cases II and III of Theorem 7-7 are left as Classroom Exercises 14 and 15. Proofs of the following three corollaries will be considered in Classroom Exercises 3-5.

Corollary 1

If two inscribed angles intercept the same arc, then the angles are congruent.

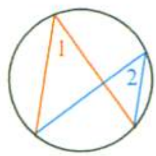
Corollary 2

If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

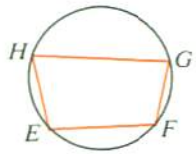
Corollary 3

An angle inscribed in a semicircle is a right angle.

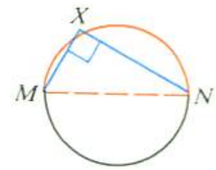
The following diagrams illustrate Corollaries 1–3.



Corollary 1
$\angle 1 \cong \angle 2$

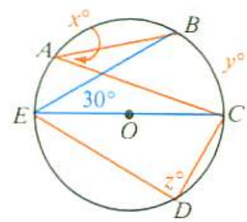


Corollary 2
$\angle E$ is supp. to $\angle G$.
$\angle F$ is supp. to $\angle H$.



Corollary 3
If $\widehat{MXN}$ is a semicircle, then $\angle X$ is a right angle.

Example Find the values of x, y, and z in $\odot O$.

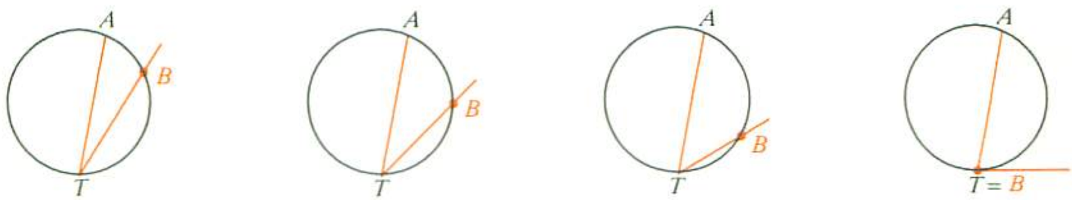


Solution $\angle BAC \cong \angle BEC$ since they intercept the same arc, so $x = 30$.

$30 = \frac{1}{2}m\widehat{BC}$, so $y = 60$.

$\angle EDC$ is inscribed in a semicircle, so $z = 90$.

Study the diagrams below from left to right. Point B moves along the circle closer and closer to point T. Finally, in diagram (4), point B has merged with T, and one side of $\angle T$ has become a tangent.



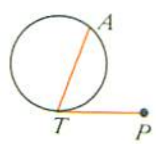
(1) (2) (3) (4)

Apply Theorem 7-7 to diagrams (1), (2), and (3) and you have $m\angle T = \frac{1}{2}m\widehat{AB}$. As you might expect, this equation applies to diagram (4), too. Diagram (4) suggests Theorem 7-8. In Exercises 9–11 you will prove the three cases of the theorem.

Theorem 7-8

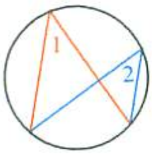
The measure of an angle formed by a chord and a tangent is equal to half the measure of the intercepted arc.

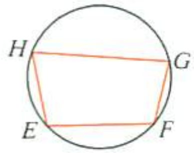
For example, if $\overline{PT}$ is tangent to the circle and $m\widehat{AT} = 140$, then $m\angle ATP = 70$.

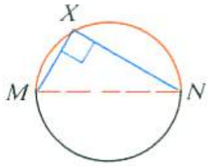


Classroom Exercises

1. A regular hexagon is inscribed in a circle. What is the measure of each arc?
2. A regular 15-gon is inscribed in a circle. What is the measure of each arc?
3. Explain why Corollary 1 of Theorem 7-7 is true. That is, explain why $\angle 1 \cong \angle 2$.



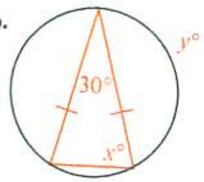




4. Explain why Corollary 2 is true. That is, explain why $\angle E$ and $\angle G$ are supplementary. (*Hint:* Let $m\widehat{FGH} = n$. Express $m\angle E$ and $m\angle G$ in terms of n.)
5. Explain why Corollary 3 is true. That is, explain how the fact that $\widehat{MXN}$ is a semicircle leads to the conclusion that $\angle X$ is a right angle.

Find the values of x and y. In Exercise 8, O is the center of the circle.

6.



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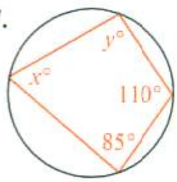

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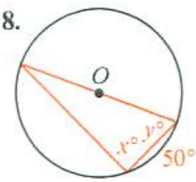
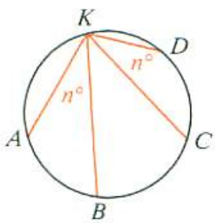
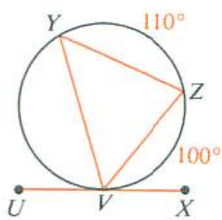


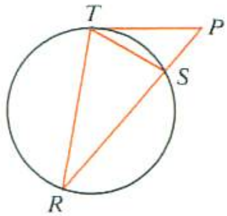
9. In quadrilateral $RSTU$, it is known that $m\angle R = 80$, $m\angle S = 100$, $m\angle T = 110$, and $m\angle U = 70$. Explain how you know that it isn't possible to draw a circle through points R, S, T, and U.
10. Suppose a parallelogram is inscribed in a circle. What special kind of parallelogram must it be? Explain.
11. In the figure, $m\angle AKB = m\angle CKD = n$. $m\widehat{AB} = $ _?_ and $m\widehat{CD} = $ _?_. State a theorem suggested by this exercise.



Ex. 11



Ex. 12



Ex. 13

12. $\overline{VX}$ is tangent to the circle. State the degree measures of as many angles as possible.
13. $\overline{PT}$ is tangent to the circle. Compare the measures of $\angle PRT$ and $\angle PTS$.

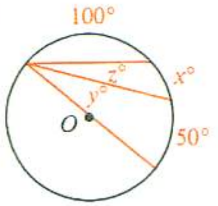
14. Outline a proof of Case II of Theorem 7-7. Use the diagram on page 312. (*Hint:* Draw the diameter from B and apply Case I.)

15. Repeat Exercise 14 for Case III.

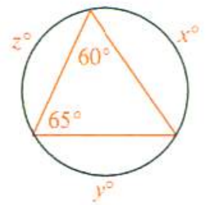
Written Exercises

When the letter O is used in a diagram in these exercises, point O is the center of the circle. In Exercises 1-6, find the values of x, y, and z.

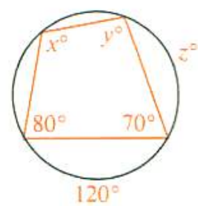
A 1.



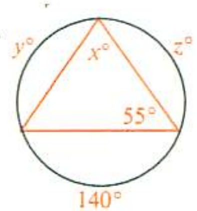
2.



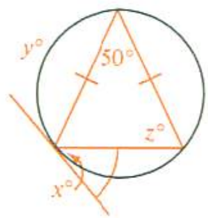
3.



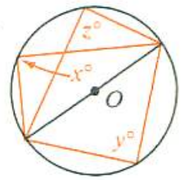
4.



5.



6.

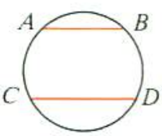


7. Prove: If two chords of a circle are parallel, the two arcs between the chords are congruent.

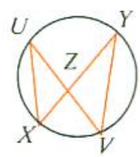
   Given: $\overline{AB} \parallel \overline{CD}$

   Prove: $\widehat{AC} \cong \widehat{BD}$

   (*Hint:* Draw $\overline{BC}$.)



8. Prove: $\triangle UXZ \sim \triangle YVZ$

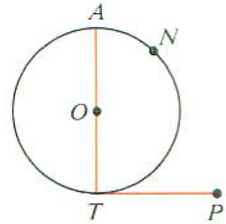


Exercises 9-11 prove the three possible cases of Theorem 7-8. In each case you are given chord $\overline{TA}$ and tangent $\overline{TP}$ of $\odot O$.

9. Supply reasons for the key steps of the proof that $m\angle ATP = \frac{1}{2}m\widehat{ANT}$ in Case I.

   Case I: O lies on $\angle ATP$.

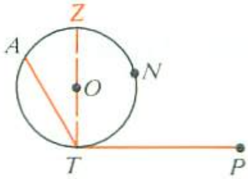
   1. $\overline{TP} \perp \overline{TA}$ and $m\angle ATP = 90$.
   2. $\widehat{ANT}$ is a semicircle and $m\widehat{ANT} = 180$.
   3. $\frac{1}{2}m\widehat{ANT} = 90$
   4. $m\angle ATP = \frac{1}{2}m\widehat{ANT}$



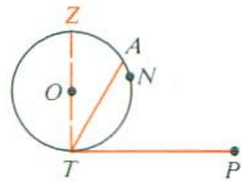
In Case II and Case III, $\overline{AT}$ is not a diameter. You can draw diameter $\overline{TZ}$ and then use Case I, Theorem 7-7, and the Angle Addition and Arc Addition Postulates.

B **10.** Prove $m\angle ATP = \frac{1}{2}m\widehat{ANT}$ in Case II.

11. Prove $m\angle ATP = \frac{1}{2}m\widehat{ANT}$ in Case III.



Case II. O lies inside $\angle ATP$.



Case III. O lies outside $\angle ATP$.

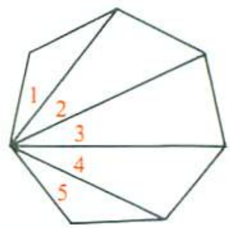
12. Points A, B, C, D, and E are five consecutive vertices of a regular inscribed 15-gon. Chord $\overline{BE}$ is drawn. $m\angle ABE =$ _?_

In Exercises 13 and 14, quadrilateral *FGHJ* is inscribed in a circle. Give numerical answers.

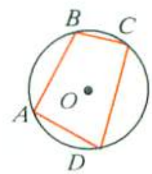
13. $m\angle F = x$, $m\angle G = x$, and $m\angle H = x + 20$. $m\angle J =$ _?_

14. $m\angle F = x^2$, $m\angle G = 9x - 2$, $m\angle H = 11x$, and $m\angle J = x^2 + 20$. The measure of the largest angle of the quadrilateral is _?_.

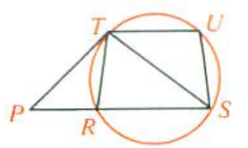
15. The diagram at the right shows a regular polygon with 7 sides.
a. Explain why the numbered angles are all congruent. (*Hint:* You may assume that a circle can be circumscribed about any regular polygon.)
b. Will your reasoning apply to a regular polygon with any number of sides?



C **16.** Given: Vertices A, B, and C of quadrilateral $ABCD$ lie on $\odot O$; $m\angle A + m\angle C = 180$; $m\angle B + m\angle D = 180$.
Prove: D lies on $\odot O$.
(*Hint:* Use an indirect proof. Assume temporarily that D is not on $\odot O$. You must then treat two cases: (1) D is inside $\odot O$, and (2) D is outside $\odot O$. In each case let X be the point where $\overrightarrow{AD}$ intersects $\odot O$ and draw $\overline{CX}$. Show that what you can conclude about $\angle AXC$ contradicts the given information.)

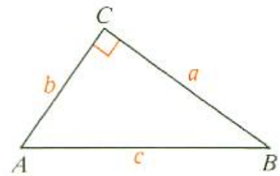


17. Given: $\overline{PT}$ is a tangent; $\overline{TU} \parallel \overline{PS}$.
Find three similar triangles and prove them similar. Write a paragraph proof.



18. Circle I is inscribed in $\triangle FGH$ and $\odot O$ is circumscribed about $\triangle FGH$. $\overrightarrow{FI}$ intersects $\odot O$ in a point K. Discover and prove a relationship between $\overline{KG}$, $\overline{KH}$, and $\overline{KI}$.

★ **19.** Angle C of $\triangle ABC$ is a right angle. The sides of the triangle have the lengths shown. The smallest circle (not shown) through C that is tangent to $\overline{AB}$ intersects $\overline{AC}$ at J and $\overline{BC}$ at K. Express the distance JK in terms of a, b, and c.



★ **20.** Prove: When a circle is circumscribed about an equilateral triangle, and chords are drawn from any point on the circle to the three vertices of the triangle, then the length of the longest chord is equal to the sum of the lengths of the other two chords.

★ **21.** Prove: The product of the lengths of the diagonals of an inscribed quadrilateral is equal to the sum of the products of the lengths of the opposite sides. (*Hint:* If the quadrilateral is $ABCD$, draw a segment, intersecting $\overline{AC}$ at X, such that $\angle ADX \cong \angle BDC$.)

7-6 Other Angles

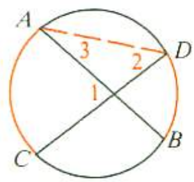
The preceding section dealt with angles that have their vertices on a circle. Theorem 7-9 deals with the angle formed by two chords that intersect inside a circle. Such an angle and its vertical angle intercept two arcs.

Theorem 7-9

The measure of an angle formed by two chords that intersect inside a circle is equal to half the sum of the measures of the intercepted arcs.

Given: Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle.

Prove: $m\angle 1 = \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$



Proof:

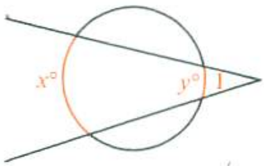
Statements	Reasons
1. Draw chord $\overline{AD}$.	1. Through any two points there is exactly one line.
2. $m\angle 1 = m\angle 2 + m\angle 3$	2. The measure of an exterior $\angle$ of a $\triangle$ = the sum of the measures of the two remote interior $\angle$s.
3. $m\angle 2 = \frac{1}{2}m\widehat{AC}$; $m\angle 3 = \frac{1}{2}m\widehat{BD}$	3. The measure of an inscribed angle is equal to half the measure of its intercepted arc.
4. $m\angle 1 = \frac{1}{2}m\widehat{AC} + \frac{1}{2}m\widehat{BD}$, or $m\angle 1 = \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$	4. Substitution (Step 3 in Step 2)

One case of the next theorem will be proved in Classroom Exercise 10, the other two cases in Exercises 22 and 23.

Theorem 7-10

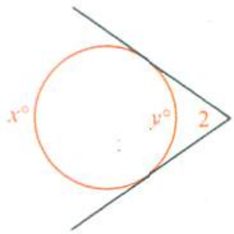
The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside a circle is equal to half the difference of the measures of the intercepted arcs.

Case I: Two secants



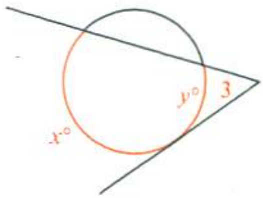
$m\angle 1 = \frac{1}{2}(x - y)$

Case II: Two tangents



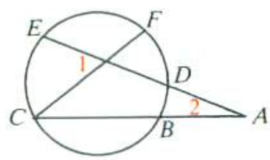
$m\angle 2 = \frac{1}{2}(x - y)$

Case III: A secant and a tangent



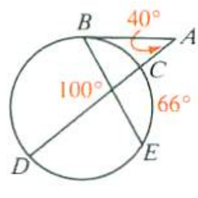
$m\angle 3 = \frac{1}{2}(x - y)$

Example 1 $m\widehat{CE} = 70$, $m\widehat{FD} = 50$, and $m\widehat{DB} = 26$. Find the measures of $\angle 1$ and $\angle 2$.



Solution
$m\angle 1 = \frac{1}{2}(m\widehat{CE} + m\widehat{FD})$ (Theorem 7-9)
$m\angle 1 = \frac{1}{2}(70 + 50) = 60$
$m\angle 2 = \frac{1}{2}(m\widehat{CE} - m\widehat{BD})$ (Theorem 7-10)
$m\angle 2 = \frac{1}{2}(70 - 26) = 22$

Example 2 $\overline{BA}$ is a tangent. Find $m\widehat{BD}$ and $m\widehat{BC}$.

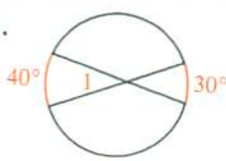


Solution
$100 = \frac{1}{2}(m\widehat{BD} + 66)$ (Theorem 7-9)
$200 = m\widehat{BD} + 66$, so $m\widehat{BD} = 134$
$40 = \frac{1}{2}(m\widehat{BD} - m\widehat{BC})$ (Theorem 7-10)
$80 = 134 - m\widehat{BC}$, so $m\widehat{BC} = 54$

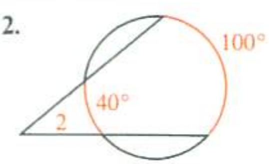
Classroom Exercises

Find the measure of each numbered angle.

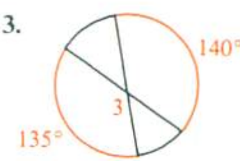
1.



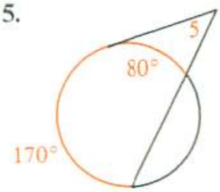
2.

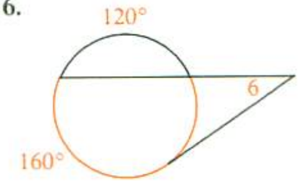


3.

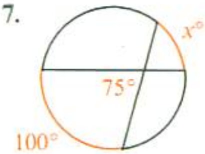


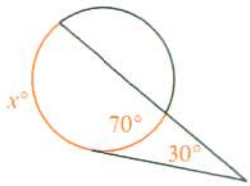
4. 

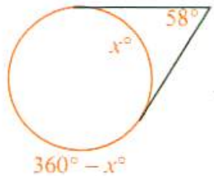
5. 

6. 

State an equation you can use to find x. Then find the value of x.

7. 

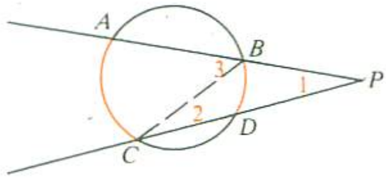
8. 

9. 

10. Supply reasons to complete a proof of the first case of Theorem 7-10.

Given: Secants $\overline{PA}$ and $\overline{PC}$

Prove: $m\angle 1 = \frac{1}{2}(m\widehat{AC} - m\widehat{BD})$

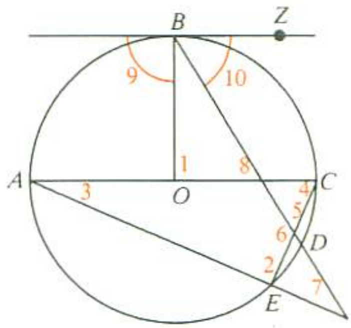


Proof:

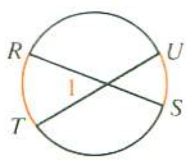
1. Draw chord $\overline{BC}$.
2. $m\angle 1 + m\angle 2 = m\angle 3$
3. $m\angle 1 = m\angle 3 - m\angle 2$
4. $m\angle 3 = \frac{1}{2}m\widehat{AC}$; $m\angle 2 = \frac{1}{2}m\widehat{BD}$
5. $m\angle 1 = \frac{1}{2}m\widehat{AC} - \frac{1}{2}m\widehat{BD}$, or
   $m\angle 1 = \frac{1}{2}(m\widehat{AC} - m\widehat{BD})$

Written Exercises

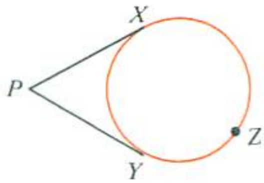
A **1-10.** $\overleftrightarrow{BZ}$ is tangent to $\odot O$; $\overline{AC}$ is a diameter; $m\widehat{BC} = 90$; $m\widehat{CD} = 30$; $m\widehat{DE} = 20$
Draw your own large diagram so that you can write arc measures alongside the arcs. Find the measure of each numbered angle.



11. If $m\widehat{RT} = 80$ and $m\widehat{US} = 40$, then $m\angle 1 =$ __?__.
12. If $m\widehat{RU} = 130$ and $m\widehat{TS} = 100$, then $m\angle 1 =$ __?__.
13. If $m\angle 1 = 50$ and $m\widehat{RT} = 70$, then $m\widehat{US} =$ __?__.
14. If $m\angle 1 = 52$ and $m\widehat{US} = 36$, then $m\widehat{RT} =$ __?__.

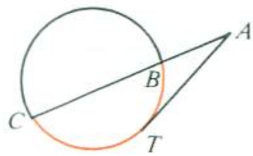


In Exercises 15–17, $\overline{PX}$ and $\overline{PY}$ are tangents.



15. If $m\widehat{XZY} = 250$, then $m\angle P = \underline{\ ?\ }$.
16. If $m\widehat{XY} = 90$, then $m\angle P = \underline{\ ?\ }$.
17. If $m\angle P = 85$, then $m\widehat{XY} = \underline{\ ?\ }$.

In Exercises 18–20, $\overline{AT}$ is a tangent.



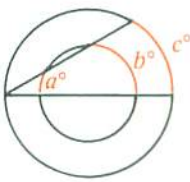
18. If $m\widehat{CT} = 110$ and $m\widehat{BT} = 50$, then $m\angle A = \underline{\ ?\ }$.
19. If $m\angle A = 40$ and $m\widehat{BT} = 40$, then $m\widehat{CT} = \underline{\ ?\ }$.
20. If $m\angle A = 35$ and $m\widehat{CT} = 110$, then $m\widehat{BT} = \underline{\ ?\ }$.

B 21. A quadrilateral circumscribed about a circle has angles of 80°, 90°, 94°, and 96°. Find the measures of the four nonoverlapping arcs determined by the points of tangency.

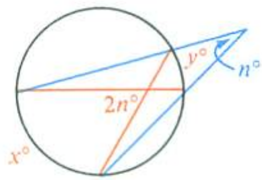
22. Prove Case II of Theorem 7-10. (*Hint:* See Classroom Exercise 10. In a figure like the second one shown below the theorem on page 318, draw the chord joining the points of tangency.)

23. Prove Case III of Theorem 7-10.

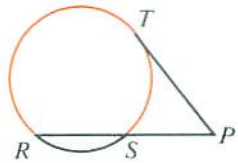
24. Write an equation involving a, b, and c.



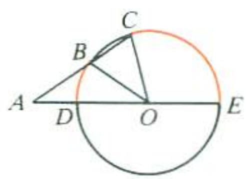
25. Find the ratio $x:y$.



C 26. $\overline{PT}$ is a tangent. It is known that $80 < m\widehat{RS} < m\widehat{ST} < 90$. State as much as you can about the measure of $\angle P$.



27. $\overline{AC}$ and $\overline{AE}$ are secants of $\odot O$. It is given that $\overline{AB} \cong \overline{OB}$. Discover and prove a relation between the measures of $\widehat{CE}$ and $\widehat{BD}$.



28. Take any point P outside a circle. Draw a tangent segment $\overline{PT}$ and a secant $\overline{PBA}$ with A and B points on the circle. Take K on $\overline{PA}$ so that $PK = PT$. Draw $\overrightarrow{TK}$. Let the intersection of $\overrightarrow{TK}$ with the circle be point X. Discover and prove a relationship between $\widehat{AX}$ and $\widehat{XB}$.

7-7 Circles and Lengths of Segments

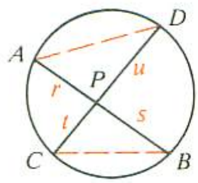
You can use similar triangles to prove that lengths of chords, secants, and tangents are related in interesting ways.

Theorem 7-11

When two chords intersect inside a circle, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord.

Given: $\overline{AB}$ and $\overline{CD}$ intersect in P.

Prove: $r \cdot s = t \cdot u$



Proof:

Statements	Reasons
1. Draw chords $\overline{AD}$ and $\overline{CB}$.	1. Through any two points there is exactly one line.
2. $\angle A \cong \angle C$; $\angle D \cong \angle B$	2. If two inscribed angles intercept __?__.
3. $\triangle APD \sim \triangle CPB$	3. Why?
4. $\frac{r}{t} = \frac{u}{s}$	4. Why?
5. $r \cdot s = t \cdot u$	5. A property of proportions

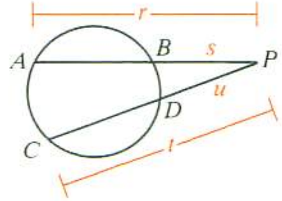
For a proof of the following theorem, see Classroom Exercise 7. In the diagram for the theorem, $\overline{AP}$ and $\overline{CP}$ are *secant segments*. $\overline{BP}$ and $\overline{DP}$ are exterior to the circle and are referred to as *external segments*.

Theorem 7-12

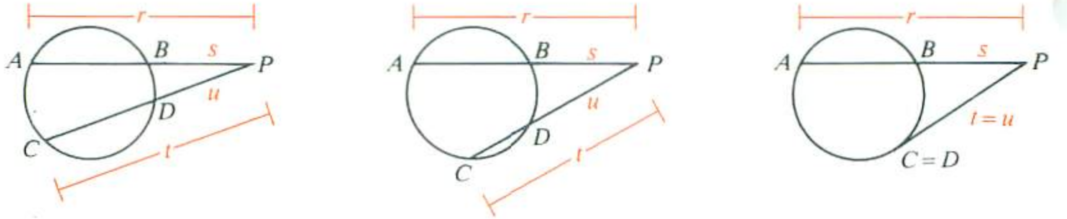
When two secant segments are drawn to a circle from an external point, the product of the lengths of one secant segment and its external segment equals the product of the lengths of the other secant segment and its external segment.

Given: $\overline{PA}$ and $\overline{PC}$ drawn to the circle from point P

Prove: $r \cdot s = t \cdot u$



Study the diagrams below from left to right. As $\overline{PC}$ approaches a position of tangency, C and D move closer together until they merge, $\overline{PC}$ is a tangent, and $t = u$.

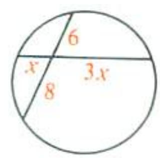


In the first two diagrams we know that $r \cdot s = t \cdot u$. In the third figure, u and t both become equal to the length of the tangent segment, and the equation becomes $r \cdot s = t^2$. This result, stated below, will be proved in Exercise 10.

Theorem 7–13

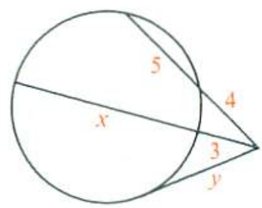
When a secant segment and a tangent segment are drawn to a circle from an external point, the product of the lengths of the secant segment and its external segment is equal to the square of the length of the tangent segment.

Example 1 Find the value of x.



Solution $3x \cdot x = 6 \cdot 8$ (Theorem 7–11)
$3x^2 = 48$, $x^2 = 16$, and $x = 4$

Example 2 Find the values of x and y.

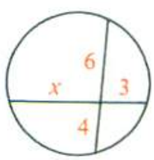


Solution $4 \cdot 9 = 3(3 + x)$ (Theorem 7–12)
$36 = 3(3 + x)$, $12 = 3 + x$, and $x = 9$
$4 \cdot 9 = y^2$ (Theorem 7–13)
$36 = y^2$, so $y = 6$

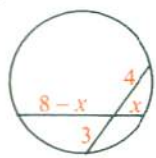
Classroom Exercises

Chords, secants, and tangents are shown. State the equation you would use to find x. Then solve for x.

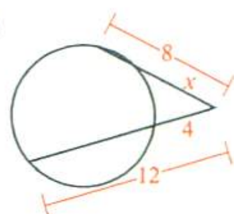
1.

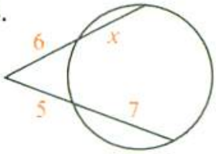


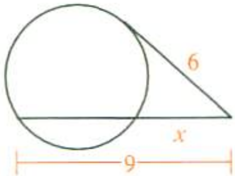
2.

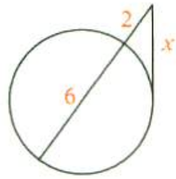


3.



4. 

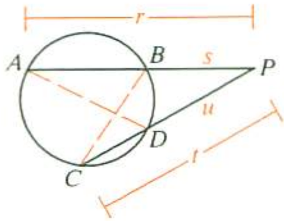
5. 

6. 

7. Supply reasons to complete the proof of Theorem 7-12.
Given: $\overline{PA}$ and $\overline{PC}$ drawn to the circle from point P
Prove: $r \cdot s = t \cdot u$

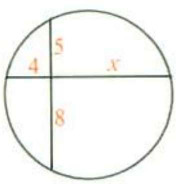
Proof:

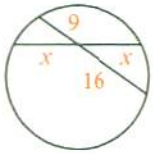
1. Draw chords $\overline{AD}$ and $\overline{BC}$.
2. $\angle A \cong \angle C$
3. $\angle P \cong \angle P$
4. $\triangle APD \sim \triangle CPB$
5. $\frac{r}{t} = \frac{u}{s}$
6. $r \cdot s = t \cdot u$

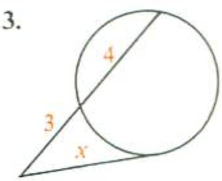


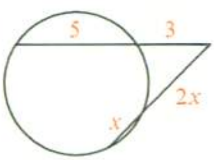
Written Exercises

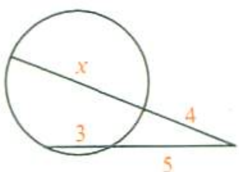
Chords, secants, and tangents are shown. Find the value of x.

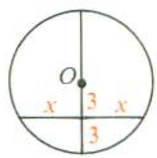
A 1. 

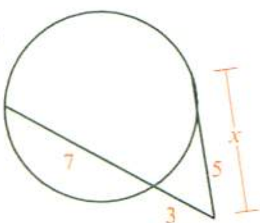
2. 

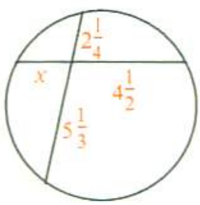
3. 

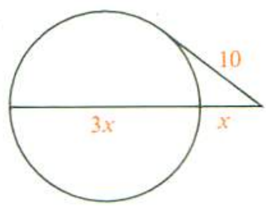
4. 

5. 

6. 

7. 

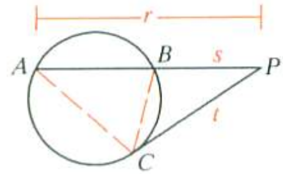
8. 

9. 

10. Copy and complete the proof of Theorem 7-13.

Given: Secant segment $\overline{PA}$ and tangent segment $\overline{PC}$ drawn to the circle from P.

Prove: $r \cdot s = t^2$



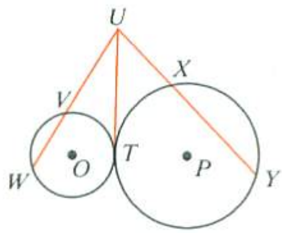
Proof:

Statements	Reasons
1. Draw chords $\overline{AC}$ and $\overline{BC}$.	1. _?_
2. $m\angle A = \frac{1}{2}m\widehat{BC}$	2. _?_
3. $m\angle BCP = \frac{1}{2}m\widehat{BC}$	3. The measure of an angle formed by a chord and a tangent _?_.
4. $\angle A \cong \angle BCP$	4. _?_
5. $\angle P \cong \angle P$	5. _?_

(*Hint:* You need three more steps. Apply similar triangles as in Classroom Exercise 7.)

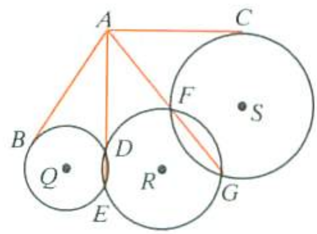
B 11. Given: $\odot O$ and $\odot P$ are tangent at T.

Prove: $UV \cdot UW = UX \cdot UY$



12. Given: $\overline{AB}$ is tangent to $\odot Q$; $\overline{AC}$ is tangent to $\odot S$.

Prove: $\overline{AB} \cong \overline{AC}$

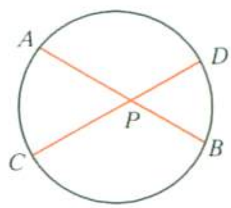


Chords $\overline{AB}$ and $\overline{CD}$ intersect at P. Find the lengths indicated.

Example: $AP = 5$; $BP = 4$; $CD = 12$; $CP =$ _?_

Solution: Let $CP = x$. Then $DP = 12 - x$.

$x(12 - x) = 5 \cdot 4$
$12x - x^2 = 20$
$x^2 - 12x + 20 = 0$
$(x - 2)(x - 10) = 0$
$x = 2$ or $x = 10$
$CP = 2$ or 10



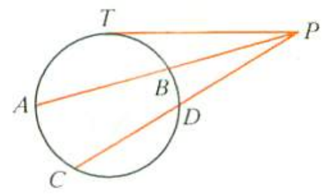
13. $AP = 6$; $BP = 8$; $CD = 16$; $DP =$ _?_

14. $CD = 10$; $CP = 6$; $AB = 11$; $AP =$ _?_

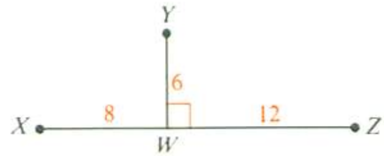
15. $AB = 12$; $CP = 9$; $DP = 4$; $BP =$ _?_

16. $AP = 6$; $BP = 5$; $CP = 3 \cdot DP$; $DP =$ _?_

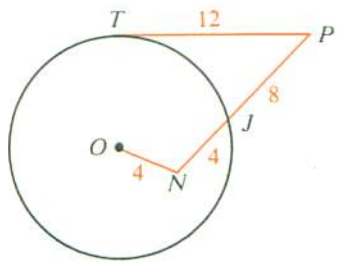
$\overline{PT}$ is tangent to the circle. Find the lengths indicated.



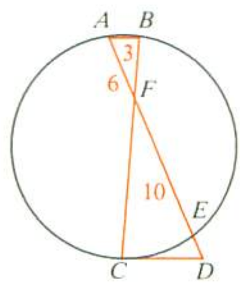
17. $PT = 6$; $PB = 3$; $AB = \underline{\ ?\ }$
18. $PT = 12$; $CD = 18$; $PC = \underline{\ ?\ }$
19. $PD = 5$; $CD = 7$; $AB = 11$; $PB = \underline{\ ?\ }$
20. $PB = AB = 5$; $PD = 4$; $PT = \underline{\ ?\ }$ and $PC = \underline{\ ?\ }$
21. A circle can be drawn through points X, Y, and Z.
    a. What is the radius of the circle?
    b. How far is the center of the circle from point W?



C 22. $\overline{PT}$ is tangent to $\odot O$ and $\overline{PN}$ intersects $\odot O$ at J. Find the radius of the circle.



Ex. 22

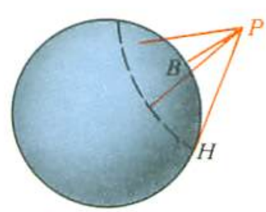


Ex. 23

★23. In the diagram at the right above, $\overline{CD}$ is a tangent, $\widehat{AC} \cong \widehat{BC}$, $AB = 3$, $AF = 6$, and $FE = 10$. Find ED.

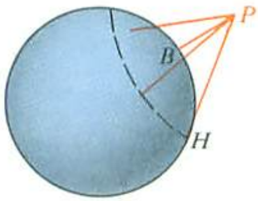
Application

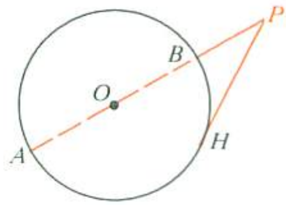
DISTANCE TO THE HORIZON



If you look out over the surface of the Earth from a position at P, directly above point B on the surface, you see the horizon wherever your line of sight is tangent to the surface of the Earth. If the surface around B is smooth (say you are on the ocean on a calm day), the horizon will be a circle, and the higher your lookout is, the farther away this horizon circle will be.

You can use Theorem 7-13 to derive a formula that tells how far you can see from any given height. As shown on the following page, the picture is simpler if you imagine a section through the Earth containing P, H, and O, the center of the Earth.





In the diagram at the right above, $\overline{PH}$ is tangent to circle O at H. $\overline{PA}$ is a secant passing through the center O. By Theorem 7-13:

$$(PH)^2 = AP \cdot BP$$

PH is the distance from the observer to the horizon, and BP is the observer's height above the surface of the Earth. If this height is small compared to the diameter, AB, of the Earth, then $AP \approx AB$. Using 12,800,000 m for AB, you can rewrite the formula as:

$$(\text{distance})^2 \approx (12{,}800{,}000)(\text{height})$$

Taking square roots, you get:

$$\text{distance} \approx \sqrt{12{,}800{,}000} \cdot \sqrt{\text{height}} \approx 3600\sqrt{\text{height}}$$

So the approximate distance (in meters) to the horizon is 3600 times the square root of your height (in meters) above the surface of the Earth. If your height is less than 400 km, the error in this approximation will be less than one percent.

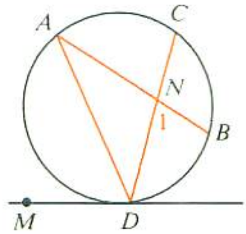
Exercises

In Exercises 1 and 2 give your answer to the nearest kilometer, in Exercises 3 and 5 to the nearest 10 km, and in Exercise 4 to the nearest meter.

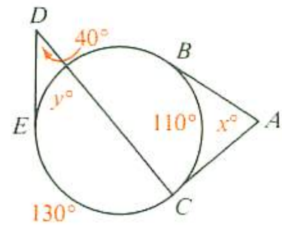
1. If you stand on a dune with your eyes about 16 m above sea level, how far out to sea can you look?
2. A lookout climbs high in the rigging of a sailing ship to a point 36 m above the water line. About how far away is the horizon?
3. From a balloon floating 10 km above the ocean, how far away is the farthest point you can see on the Earth's surface?
4. If you want to have a horizon of 8 km, how high a lookout must you have?
5. You are approaching the coast of Japan in a small sailboat. The highest point on the central island of Honshu is the cone of Mount Fuji, 3776 m above sea level. Roughly how far away from the mountain will you be when you can first see the top? (Assume that the sky is clear!)

Self-Test 2

$\overleftrightarrow{MD}$ is tangent to the circle.

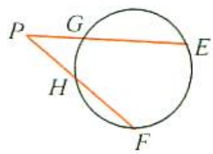


1. If $m\widehat{BD} = 80$, then $m\angle A = \underline{\ ?\ }$.
2. If $m\angle ADM = 75$, then $m\widehat{AD} = \underline{\ ?\ }$.
3. If $m\widehat{BD} = 80$ and $m\angle 1 = 81$, then $m\widehat{AC} = \underline{\ ?\ }$.
4. If $AN = 12$, $BN = 6$, and $CN = 8$, then $DN = \underline{\ ?\ }$.

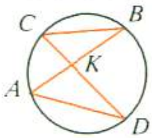


5. $\overline{AB}$, $\overline{AC}$, and $\overline{DE}$ are tangents. Find the values of x and y.

$\overline{PE}$ and $\overline{PF}$ are secants.



6. If $m\widehat{EF} = 100$ and $m\widehat{GH} = 30$, then $m\angle P = \underline{\ ?\ }$.
7. If $PG = 4$, $PE = 15$, and $PH = 6$, then $PF = \underline{\ ?\ }$.
8. If $PG = 8$, $GE = 12$, and $HF = 6$, then $PH = \underline{\ ?\ }$.
9. Given: Chords as shown.
   Prove: $\dfrac{AK}{CK} = \dfrac{AD}{CB}$



CALCULATOR KEY-IN

The ratio of the circumference of a circle to the diameter is a constant for all circles. The ratio is denoted by the Greek letter π (*pi*).

$$\pi = \frac{C}{d}$$

If you were to wrap a string around a circle to measure the circumference, C, and if you then measured the diameter, d, you would find that $\frac{C}{d} \approx 3$.

π is an irrational number that cannot be expressed exactly as the ratio of two integers. Decimal approximations of π have been computed to thousands of decimal places. (The value of π to 5 places is 3.14159.) We can easily look up values in reference books, but such was not always the case. In the past, mathematicians had to rely on their cleverness to compute an approximate value of π. One of the earliest approximations was that of Archimedes, who found that $3\frac{1}{7} > \pi > 3\frac{10}{71}$.

Exercises

1. Find decimal approximations of $3\frac{1}{7}$ and $3\frac{10}{71}$. Did Archimedes approximate π correct to hundredths?

In Exercises 2–5, find approximations for π. The more terms or factors you use, the better your approximations will be. But you can't use them all!

2. $\pi \approx 2\sqrt{3}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \frac{1}{3^5 \cdot 11} + \cdots\right)$ (Sharpe, 18th century)

3. $\pi \approx 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \ldots$ (Wallis, 17th century)

4. This exercise is for calculators that have a square root function and a memory.
$$\pi \approx 2 \div \left(\sqrt{0.5} \cdot \sqrt{0.5 + 0.5\sqrt{0.5}} \cdot \sqrt{0.5 + 0.5\sqrt{0.5 + 0.5\sqrt{0.5}}} \cdot \ldots\right)$$
(Vieta, 16th century)

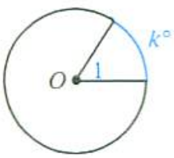
5. $\pi \approx 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)$ (Leibniz, 17th century)

Note: Although the expression here is simple in appearance, your approximations will move *very* slowly toward π. If you used one hundred terms within the parentheses, your approximation would not be correct to more than one decimal place.

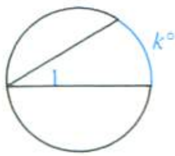
Chapter Summary

1. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency. The converse is also true.
2. Tangents to a circle from a point are congruent.
3. In the same circle or in congruent circles:
   a. Congruent minor arcs have congruent central angles.
   Congruent central angles have congruent arcs.
   b. Congruent chords have congruent arcs.
   Congruent arcs have congruent chords.
   c. Congruent chords are equally distant from the center.
   Chords equally distant from the center are congruent.
4. A diameter that is perpendicular to a chord bisects the chord and its arc.
5. If two inscribed angles intercept the same arc, then the angles are congruent.
6. If a quadrilateral is inscribed in a circle, then opposite angles are supplementary.
7. An angle inscribed in a semicircle is a right angle.

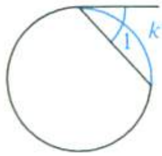
8. Relationships expressed by formulas:



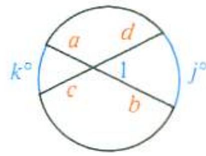
$m\angle 1 = k$



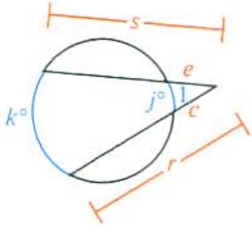
$m\angle 1 = \frac{1}{2}k$



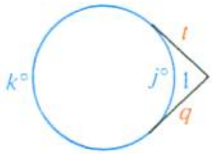
$m\angle 1 = \frac{1}{2}k$



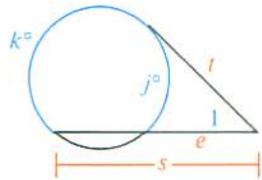
$m\angle 1 = \frac{1}{2}(k + j)$
$a \cdot b = c \cdot d$



$m\angle 1 = \frac{1}{2}(k - j)$
$s \cdot e = r \cdot c$



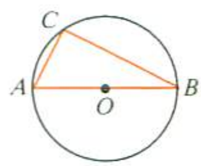
$m\angle 1 = \frac{1}{2}(k - j)$
$t = q$



$m\angle 1 = \frac{1}{2}(k - j)$
$s \cdot e = t^2$

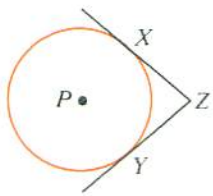
Chapter Review

Points A, B, and C lie on $\odot O$. 7-1



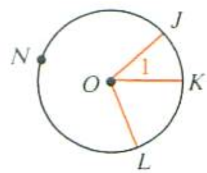
1. $\overline{AC}$ is called a __?__, while $\overleftrightarrow{AC}$ is called a __?__.
2. $\overline{OB}$ is called a __?__.
3. The best name for $\overline{AB}$ is __?__.
4. $\triangle ABC$ is ______?______ $\odot O$.
   (inscribed in/circumscribed about)

Lines $\overleftrightarrow{ZX}$ and $\overleftrightarrow{ZY}$ are tangent to $\odot P$. 7-2



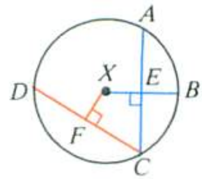
5. $\overline{PX}$, if drawn, would be __?__ to $\overleftrightarrow{XZ}$.
6. If the radius of $\odot P$ is 6 and if $XZ = 8$, the distance between points P and Z is __?__.
7. If $m\angle Z = 90$ and if $XZ = 13$, the distance between points X and Y is __?__.

7-3



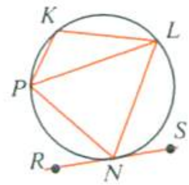
8. If $m\angle 1 = 42$, then $m\widehat{JK} =$ __?__.
9. If $m\widehat{JN} = 120$ and $m\widehat{NL} = 130$, then $m\angle JOL =$ __?__.
10. Suppose $\overrightarrow{JO}$ intersects $\odot O$ at G.
    a. $m\widehat{JLG} =$ __?__
    b. If $\angle NOG \cong \angle KOL$, then $\widehat{NG}$ __?__ $\widehat{KL}$.

In $\odot X$, $m\widehat{AC} = 120$. 7-4

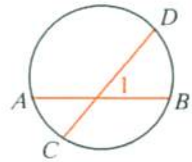


11. $m\widehat{AB} = \underline{\ ?\ }$.
12. If $\overline{AC} \cong \overline{CD}$, then $m\widehat{CD} = \underline{\ ?\ }$.
13. If $CD > AC$, then $XF \underset{(</=/>)}{\underline{\quad ?\quad}} XE$.
14. If $DC = 24$ and $XF = 5$, the radius of $\odot X = \underline{\ ?\ }$.

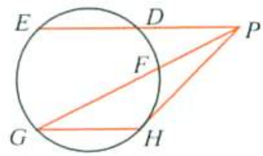
$\overleftrightarrow{RS}$ is tangent to the circle at N. 7-5



15. If $m\angle K = 105$, then $m\angle PNL = \underline{\ ?\ }$.
16. If $m\widehat{PN} = 100$, then $m\angle PLN = \underline{\ ?\ }$ and $m\angle PNR = \underline{\ ?\ }$.
17. If $m\angle K = 110$, then $m\widehat{PNL} = \underline{\ ?\ }$ and $m\widehat{PL} = \underline{\ ?\ }$.

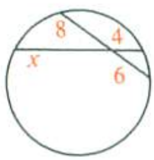


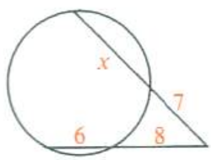
18. If $m\widehat{AC} = 40$ and $m\widehat{BD} = 60$, then $m\angle 1 = \underline{\ ?\ }$. 7-6
19. If $m\widehat{AC} = 44$ and $m\angle 1 = 55$, then $m\widehat{BD} = \underline{\ ?\ }$.

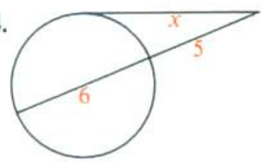


20. If $m\widehat{EG} = 100$ and $m\widehat{DF} = 40$, then $m\angle EPG = \underline{\ ?\ }$.
21. If $\overline{PH}$ is a tangent, $m\widehat{GH} = 90$ and $m\angle GPH = 25$, then $m\widehat{FH} = \underline{\ ?\ }$.

Chords, secants, and a tangent are shown. Find x. 7-7

22. 

23. 

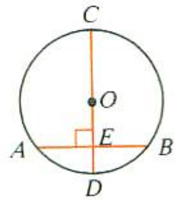
24. 

Chapter Test

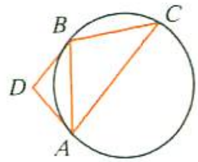
Tell whether the statement is true or false.

1. It is possible to place points A, B, and C on a circle so that $m\widehat{AB} + m\widehat{BC} > m\widehat{AC}$.
2. If two circles are congruent, their diameters are congruent.
3. If a chord in one circle is congruent to a chord in another circle, the arcs of these chords must have congruent central angles.
4. Opposite angles of an inscribed quadrilateral must be congruent.

5. If a diameter is perpendicular to a chord, the diameter must bisect the chord.
6. If a line bisects a chord, that line must pass through the center of the circle.
7. If $\overrightarrow{GM}$ intersects a circle in just one point, $\overline{GM}$ must be tangent to the circle.
8. It is possible to draw two circles so that no common tangents can be drawn.
9. An angle inscribed in a semicircle must be a right angle.
10. When one chord is farther from the center of a circle than another chord, the chord farther from the center is the longer of the two chords.

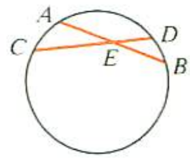


11. In $\odot O$, if $m\widehat{AB} = 100°$, then $m\widehat{AC} =$ _?_.
12. If the radius of $\odot O$ is 17 and $AB = 30$, then $OE =$ _?_.

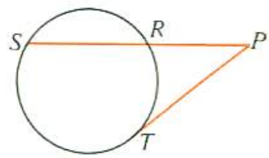


$\overline{AD}$ and $\overline{DB}$ are tangent to the circle.

13. If $AB = BC$ and $m\widehat{BC} = 80$, then $m\angle ABC =$ _?_.
14. If $m\angle D = 110$, then $m\angle BCA =$ _?_.



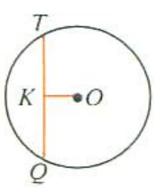
15. If $m\widehat{AC} = 50$ and $m\widehat{BD} = 38$, then $m\angle AEC =$ _?_.
16. If $AE = 10$, $EB = 9$, and $CE = 15$, then $ED =$ _?_.



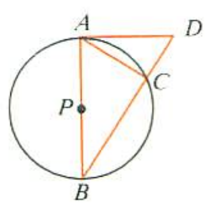
$\overline{PT}$ is a tangent to the circle.

17. If $m\widehat{RS} = 120$ and $m\widehat{ST} = 160$, then $m\angle P =$ _?_.
18. If $PT = 12$ and $PS = 18$, then $PR =$ _?_.

19. Given: $\odot O$; $\overline{TK} \cong \overline{KQ}$
Prove: $\overline{TQ} \perp \overline{OK}$



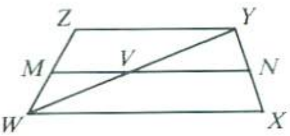
20. Given: $\overline{AD}$ is tangent to $\odot P$.
Prove: $\triangle BAD \sim \triangle ACD$



Mixed Review

1. If x, $x + 3$, and y are the lengths of the sides of a triangle, then $\underline{\ ?\ } < y < \underline{\ ?\ }$.

2. Find the measure of an angle if the measures of a supplement and a complement of the angle have the ratio 5:2.



3. Given: $\overline{MN}$ is the median of trapezoid $WXYZ$.
   Prove: $\overline{MN}$ bisects $\overline{WY}$.

4. Prove: The diagonals of a rhombus divide the rhombus into four congruent triangles.

5. A 30°-60°-90° triangle is inscribed in a circle of radius 7. Find the length of each leg of the triangle.

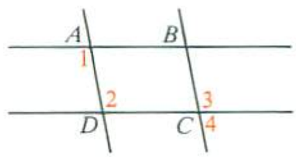
6. Must three parallel lines be coplanar? Draw a diagram to illustrate your answer.

7. If a regular polygon has 18 sides, find the measure of each interior angle and the measure of each exterior angle.

8. If $ABCD$ is a square and $AC = 4$, find AB.

9. If the lengths of two sides of a right triangle are 6 and 10, find two possible lengths for the third side.

10. Given: If Pete is particular, then Julie is jovial.
    Pete is not particular.
    What, if anything, can you conclude?



11. Given: $\angle 1 \cong \angle 2$; $\angle 2 \cong \angle 3$
    Prove: $\overline{AB} \cong \overline{DC}$

12. When the altitude to the hypotenuse of a certain right triangle is drawn, the altitude divides the hypotenuse into segments of lengths 8 and 10. Find the length of the shorter leg.

13. Complete with *outside, inside,* or *on*: In a right triangle, (a) the medians intersect $\underline{\ ?\ }$ the triangle, (b) the altitudes intersect $\underline{\ ?\ }$ the triangle, and (c) the perpendicular bisectors of the sides intersect $\underline{\ ?\ }$ the triangle.

14. If $\overrightarrow{OB}$ bisects $\angle AOC$, $m\angle AOB = 5t - 7$, and $m\angle AOC = 8t + 10$, find the numerical measure of $\angle BOC$.

15. Two chords of a circle intersect inside a circle, dividing one chord into segments of length 15 and 12 and the other chord into segments of length 9 and t. Find the value of t.

16. The sine of any acute angle must be greater than $\underline{\ ?\ }$ and less than $\underline{\ ?\ }$.

17. If points R and S on a number line have coordinates -11 and 3, and $\overline{RS}$ has midpoint T, find RS and ST.

Algebra Review

Evaluate each expression for the given values of the variables.

Example $\frac{1}{2}bh$ when $b = 3.4$ and $h = 4.5$ **Solution** $\frac{1}{2}(3.4)(4.5) = 7.65$

1. Area of a square: s^2 when $s = 1.3$
2. Length of hypotenuse of a right triangle: $\sqrt{a^2 + b^2}$ when $a = 20$ and $b = 21$
3. Perimeter of parallelogram: $2(x + y)$ when $x = \frac{5}{3}$ and $y = \frac{3}{2}$
4. Perimeter of triangle: $a + b + c$ when $a = 11.5$, $b = 7.2$, and $c = 9.9$
5. Area of a rectangle: lw when $l = 2\sqrt{6}$ and $w = 3\sqrt{3}$
6. Perimeter of isosceles trapezoid: $2r + s + t$ when $r = \frac{4}{7}$, $s = 1$, and $t = \frac{13}{7}$
7. πr^2 when $r = 30$ (Use 3.14 for π.)
8. lwh when $l = 8$, $w = 6\frac{1}{4}$, and $h = 3\frac{1}{2}$
9. $2(lw + wh + lh)$ when $l = 4.5$, $w = 3$, and $h = 1$
10. $\frac{x - 3}{y + 2}$ when $x = 3$ and $y = -4$
11. $\frac{x + 5}{y - 2}$ when $x = -2$ and $y = -4$
12. $mx + b$ when $x = -6$, $m = \frac{5}{2}$, and $b = -2$
13. $6t^2$ when $t = 3$
14. $(6t)^2$ when $t = 3$
15. $\frac{1}{2}h(a + b)$ when $h = 3$, $a = 3\sqrt{2}$, and $b = 7\sqrt{2}$
16. $\sqrt{(x - 5)^2 + (y - 3)^2}$ when $x = 1$ and $y = 0$
17. $\frac{1}{3}x^2h$ when $x = 4\sqrt{3}$ and $h = 6$
18. $2s^2 + 4sh$ when $s = \sqrt{6}$ and $h = \frac{5}{2}\sqrt{6}$

Use the given information to rewrite each expression.

Example Bh when $B = \frac{1}{2}rs$ **Solution** $Bh = \left(\frac{1}{2}rs\right)h = \frac{1}{2}rsh$

19. $c(x + y)$ when $x + y = d$
20. $\frac{1}{3}Bh$ when $B = \pi r^2$
21. $\frac{1}{2}pl$ when $p = 2\pi r$
22. $2(l + w)$ when $l = s$ and $w = s$
23. $4\pi r^2$ when $r = \frac{1}{2}d$
24. $n\left(\frac{1}{2}sa\right)$ when $ns = p$

Solve each formula for the variable shown in color.

Example $y = mx + b$ (x in color) **Solution** $x = \frac{y - b}{m}$; $m \neq 0$

25. $ax + by = c$ (x)
26. $C = \pi d$ (d)
27. $S = (n - 2)180$ (n)
28. $x^2 + y^2 = r^2$ (y)
29. $\frac{x}{h} = \frac{h}{y}$ (h)
30. $a^2 + b^2 = (a\sqrt{2})^2$ (b)
31. $A = \frac{1}{2}bh$ (h)
32. $m = \frac{y + 4}{x - 2}$ (y)

Most quilt patterns are designed by using geometric patterns. The quilt shown consists of concentric circles, squares, and triangles. The endless combinations of geometric designs used for creating patterns make quilting a beautiful and challenging craft.

Constructions and Loci

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