The Pythagorean Theorem

Objectives

1. Determine the geometric mean between two numbers.
2. State and apply the relationships that exist when the altitude is drawn to the hypotenuse of a right triangle.
3. State and apply the Pythagorean Theorem.

6-1 Geometric Means

Suppose r, s, and t are positive numbers with $\frac{r}{s} = \frac{s}{t}$. Then s is called the **geometric mean** between r and t.

Example 1 Find the geometric mean between the given numbers.

a. 3 and 7 **b.** 6 and 15

Solution

a. $\frac{3}{x} = \frac{x}{7}$

$x^2 = 21$

$x = \sqrt{21}$

b. $\frac{6}{x} = \frac{x}{15}$

$x^2 = 90$

$x = \sqrt{90} = \sqrt{9 \cdot 10} = \sqrt{9} \cdot \sqrt{10} = 3\sqrt{10}$

The symbol $\sqrt{}$ always indicates the positive square root of a number. In (b) above, the *radical* $\sqrt{90}$ could be *simplified* because the *radicand* 90 has the factor 9, a perfect square. When you write radical expressions you should write them in **simplest form.** This means writing them so that

1. No radicand has a factor, other than 1, that is a perfect square.
2. No radicand is a fraction.
3. No fraction has a denominator that contains a radical.

You should express answers involving radicals in simplest form unless you are asked to use decimal approximations.

Example 2 Simplify the expressions.

a. $3\sqrt{\frac{2}{5}}$ **b.** $\frac{5}{6\sqrt{7}}$

Solution

a. Multiply the numerator and denominator by 5 to obtain a perfect square in the denominator.

$$3\sqrt{\frac{2}{5}} = 3\sqrt{\frac{2}{5} \cdot \frac{5}{5}}$$

$$= 3\sqrt{\frac{10}{25}} = \frac{3\sqrt{10}}{5}$$

b. To remove the radical from the denominator, multiply the numerator and denominator by the radical.

$$\frac{5}{6\sqrt{7}} = \frac{5}{6\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$

$$= \frac{5\sqrt{7}}{6 \cdot 7} = \frac{5\sqrt{7}}{42}$$

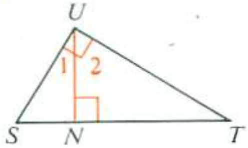
The following theorem states a special property of right triangles.

Theorem 6-1

If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.

Given: $\triangle SUT$ with rt. $\angle SUT$;
$\overline{UN} \perp \overline{ST}$

Prove: $\triangle SNU \sim \triangle SUT \sim \triangle UNT$



■ ***Plan for Proof:*** Note that $\angle 2$ and $\angle S$ are both complementary to $\angle 1$. So $\angle S \cong \angle S \cong \angle 2$. Also, $\angle SNU \cong \angle SUT \cong \angle UNT$. Therefore, $\triangle SNU \sim \triangle SUT \sim \triangle UNT$.

The altitude to the hypotenuse divides the hypotenuse into two segments. Corollaries 1 and 2 of Theorem 6-1 deal with geometric means involving the lengths of these segments. For proofs of the corollaries, see Classroom Exercises 2-7.

For simplicity in stating theorems about the lengths of sides of triangles, the words *segment, side, leg,* and *hypotenuse* are often used to refer to the *length* of a segment rather than the segment itself. When the context makes this meaning clear, as in the corollaries that follow, we will use this convention.

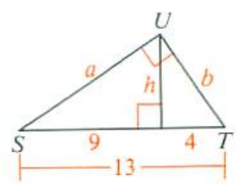
Corollary 1

When the altitude is drawn to the hypotenuse of a right triangle, the length of the altitude is the geometric mean between the segments of the hypotenuse.

Corollary 2

When the altitude is drawn to the hypotenuse of a right triangle, each leg is the geometric mean between the hypotenuse and the segment of the hypotenuse that is adjacent to that leg.

Example 3 $\angle SUT$ is a right angle. Find the values of h, a, and b.



Solution

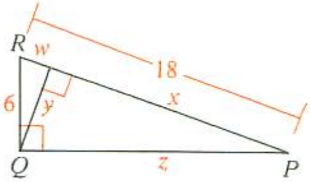
By Corollary 1:

$$\frac{9}{h} = \frac{h}{4}$$
$$h^2 = 9 \cdot 4 = 36$$
$$h = \sqrt{36}$$
$$h = 6$$

By Corollary 2:

$$\frac{13}{a} = \frac{a}{9} \qquad \frac{13}{b} = \frac{b}{4}$$
$$a^2 = 9 \cdot 13 \qquad b^2 = 4 \cdot 13$$
$$a = \sqrt{9 \cdot 13} \qquad b = \sqrt{4 \cdot 13}$$
$$a = 3\sqrt{13} \qquad b = 2\sqrt{13}$$

Example 4 $\angle PQR$ is a right angle, $RP = 18$, and $RQ = 6$.
Find the values of w, x, y, and z.



Solution First use Corollary 2 to find w.

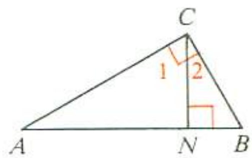
$$\frac{18}{6} = \frac{6}{w}$$
$$18w = 36$$
$$w = 2$$
Then $x = 18 - 2 = 16$.

$$\frac{16}{y} = \frac{y}{2} \text{ (Cor. 1)}$$
$$y^2 = 16 \cdot 2$$
$$y = \sqrt{16 \cdot 2}$$
$$y = 4\sqrt{2}$$

$$\frac{18}{z} = \frac{z}{16} \text{ (Cor. 2)}$$
$$z^2 = 16 \cdot 18$$
$$z = \sqrt{16 \cdot 18}$$
$$z = \sqrt{16 \cdot 9 \cdot 2}$$
$$z = 4 \cdot 3\sqrt{2} = 12\sqrt{2}$$

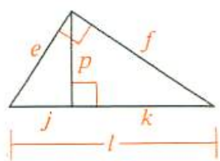
Classroom Exercises

$\angle ACB$ is a right angle and $\overline{CN} \perp \overline{AB}$.



1. If $m\angle 1 = 60$, then $m\angle A =$ _?_, $m\angle 2 =$ _?_, and $m\angle B =$ _?_.
2. If $m\angle 1 = k$, then $m\angle A =$ _?_, $m\angle 2 =$ _?_, and $m\angle B =$ _?_.
3. From Exercise 2 we see that $\angle A \cong$ _?_. Likewise, $\angle B \cong$ _?_.
4. Since $\triangle ANC \sim \triangle CNB$ (Why?), $\frac{AN}{CN} = \frac{CN}{?}$.
5. State in words the corollary proved in Exercises 2–4.
6. a. Since $\triangle ACB \sim \triangle ANC$, $\frac{AB}{AC} = \frac{AC}{?}$.

   b. Since $\triangle ACB \sim \triangle CNB$, $\frac{BA}{?} = \frac{?}{BN}$.
7. State in words the corollary proved in Exercise 6.
8. The diagram shows a right triangle with the altitude drawn to the hypotenuse.
   a. p is the geometric mean between _?_ and _?_.
   b. e is the geometric mean between _?_ and _?_.
   c. f is the geometric mean between _?_ and _?_.



9. Finish simplifying $\sqrt{\frac{7}{50}}$ in the two ways shown.

$$\sqrt{\frac{7}{50}} = \sqrt{\frac{7}{50} \cdot \frac{50}{50}} = \frac{1}{50}\sqrt{7 \cdot 2 \cdot 25} = \frac{?}{?} = \frac{?}{?}$$

$$\sqrt{\frac{7}{50}} = \sqrt{\frac{7}{50} \cdot \frac{2}{2}} = \sqrt{\frac{14}{100}} = \frac{?}{?}$$

10. Finish simplifying $\frac{2}{\sqrt{8}}$ in the two ways shown.

$$\frac{2}{\sqrt{8}} = \frac{2}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{2\sqrt{8}}{8} = \frac{?}{?} = \frac{?}{?}$$

$$\frac{2}{\sqrt{8}} = \frac{2}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{16}} = \frac{?}{?} = \frac{?}{?}$$

Simplify the radical expressions.

11. $6\sqrt{25}$ **12.** $5\sqrt{18}$ **13.** $\sqrt{\frac{1}{3}}$ **14.** $\frac{15}{\sqrt{3}}$

15. a. $\sqrt{4} \cdot \sqrt{9}$ **16. a.** $\sqrt{7} \cdot \sqrt{16}$ **17. a.** $\frac{\sqrt{4}}{\sqrt{9}}$ **18. a.** $\sqrt{\frac{5}{2}}$

b. $\sqrt{4 \cdot 9}$ **b.** $\sqrt{7 \cdot 16}$ **b.** $\sqrt{\frac{4}{9}}$ **b.** $\frac{\sqrt{5}}{\sqrt{2}}$

Written Exercises

Simplify the expressions.

A

1. $\sqrt{49}$ **2.** $3\sqrt{64}$ **3.** $\frac{2}{5}\sqrt{9}$ **4.** $\frac{2}{5}\sqrt{25}$

5. $\sqrt{12}$ **6.** $\sqrt{50}$ **7.** $5\sqrt{28}$ **8.** $\frac{1}{2}\sqrt{300}$

9. $\sqrt{\frac{1}{2}}$ **10.** $\frac{1}{\sqrt{2}}$ **11.** $\sqrt{\frac{2}{27}}$ **12.** $6\sqrt{\frac{1}{3}}$

13. $\frac{18}{\sqrt{3}}$ **14.** $\frac{15}{\sqrt{30}}$ **15.** $\frac{3\sqrt{32}}{4}$ **16.** $\frac{5}{2\sqrt{10}}$

Find the geometric mean between the two numbers.

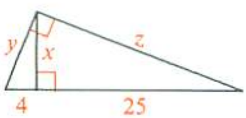
17. 2 and 8 **18.** 3 and 27 **19.** 13 and 25

20. 1 and 50 **21.** 6 and 10 **22.** $\frac{1}{10}$ and 2

Each diagram shows a right triangle with the altitude drawn to the hypotenuse. Find the values of x, y, and z.

B

23.



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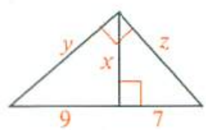

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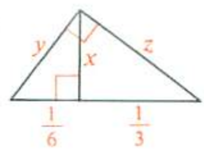

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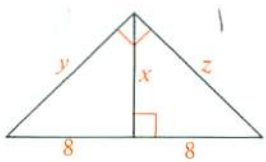

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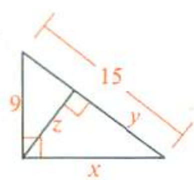

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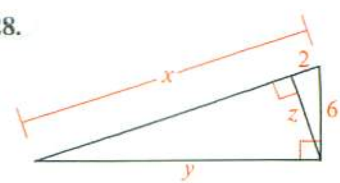
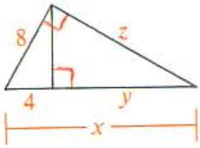
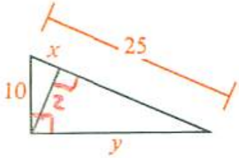
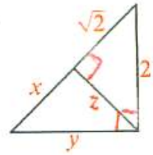


Find the values of x, y, and z.

29. 

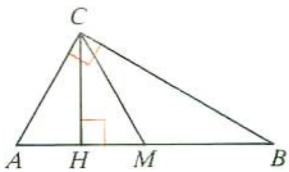
30. 

31. 

32. Prove Theorem 6-1.

C **33.** Prove: In a right triangle, the product of the hypotenuse and the length of the altitude drawn to the hypotenuse is equal to the product of the two legs.

34. The *arithmetic mean* between two numbers r and s is defined to be the number $\frac{r+s}{2}$.



a. $\overline{CM}$ is the median and $\overline{CH}$ is the altitude to the hypotenuse of right $\triangle ABC$. Show that CM is the arithmetic mean between AH and BH and that CH is the geometric mean between AH and BH. Then use the diagram to show that the arithmetic mean is greater than the geometric mean.

b. Show algebraically that the arithmetic mean between two different numbers r and s is greater than the geometric mean. (*Hint:* The geometric mean is $\sqrt{rs}$. Work backward from $\frac{r+s}{2} > \sqrt{rs}$ to $(r-s)^2 > 0$ and then reverse the steps.)

35. In this exercise p, q, r, s, and t are prime numbers, all different.

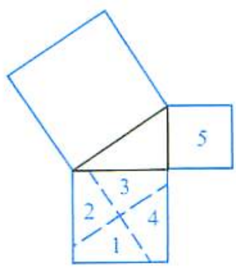
a. Note that the number 3 is the geometric mean between two different pairs of integers: 1 and 9, 3 and 3. The number 6 is the geometric mean between five different pairs of integers. List them.

b. The number pq is the geometric mean between five different pairs of integers. List them.

c. The number pqr is the geometric mean between _?_ different pairs of integers. List them.

d. The number $pqrst$ is the geometric mean between _?_ different pairs of integers. (You don't have to list them.)

Challenge



Start with a right triangle. Build a square on each side. Locate the center of the square on the longer leg. Through the center, draw a parallel to the hypotenuse and a perpendicular to the hypotenuse.

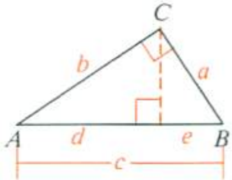
Cut out the pieces numbered 1–5. Can you arrange the five pieces to cover exactly the square built on the hypotenuse?

6-2 The Pythagorean Theorem

One of the best known and most useful theorems in all of mathematics is the Pythagorean Theorem. It is believed that Pythagoras, a Greek mathematician and philosopher, proved this theorem about twenty-five hundred years ago. Many different proofs have been discovered since then.

Theorem 6-2 Pythagorean Theorem

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.



Given: Rt. $\triangle ABC$; $\angle C$ is a rt. $\angle$.

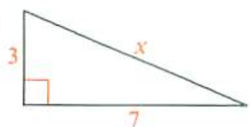
Prove: $c^2 = a^2 + b^2$

Proof:

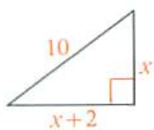
Statements	Reasons
1. Draw a perpendicular from C to $\overline{AB}$.	1. Through a point outside a line, there is exactly one line _?_.
2. $\frac{c}{a} = \frac{a}{e}$; $\frac{c}{b} = \frac{b}{d}$	2. When the altitude is drawn to the hypotenuse of a rt. $\triangle$, each leg is the geometric mean between _?_.
3. $ce = a^2$; $cd = b^2$	3. A property of proportions
4. $ce + cd = a^2 + b^2$	4. Addition Property of =
5. $c(e + d) = a^2 + b^2$	5. Distributive Property
6. $c^2 = a^2 + b^2$	6. Substitution Property

Example Find the value of x. Keep in mind the fact that the length of a segment must be a positive number.

a.



b.



Solution **a.**

$$x^2 = 7^2 + 3^2$$
$$x^2 = 49 + 9$$
$$x^2 = 58$$
$$x = \sqrt{58}$$

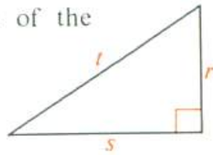
b.

$$x^2 + (x + 2)^2 = 10^2$$
$$x^2 + x^2 + 4x + 4 = 100$$
$$2x^2 + 4x - 96 = 0$$
$$x^2 + 2x - 48 = 0$$
$$(x + 8)(x - 6) = 0$$
$$\cancel{x = -8},\ x = 6$$

Classroom Exercises

1. The early Greeks thought of the Pythagorean Theorem in this form: *The area of the square on the hypotenuse of a right triangle is equal to the sum of the areas of the squares on the legs.* Draw a diagram to illustrate that interpretation.

2. For the right triangle shown, $t^2 = r^2 + s^2$. Which of the following equations are equivalent to this equation?



   a. $r^2 = s^2 + t^2$
   b. $r^2 = (t + s)(t - s)$
   c. $s^2 = t^2 - r^2$
   d. $t = r + s$

Complete each simplification.

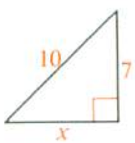
3. $(\sqrt{3})^2 = \sqrt{3} \cdot \underline{\ ?\ } = \underline{\ ?\ }$
4. $(3\sqrt{11})^2 = \underline{\ ?\ } \cdot \underline{\ ?\ } = 9 \cdot \underline{\ ?\ } = \underline{\ ?\ }$

Simplify each expression.

5. $(\sqrt{5})^2$ 6. $(2\sqrt{7})^2$ 7. $(7\sqrt{2})^2$ 8. $\left(\frac{\sqrt{2}}{2}\right)^2$

9. $\left(\frac{3}{\sqrt{5}}\right)^2$ 10. $(2n)^2$ 11. $\left(\frac{n}{\sqrt{3}}\right)^2$ 12. $\left(\frac{2}{3}\sqrt{6}\right)^2$

State an equation you could use to find the value of x. Then find the value of x.

13.


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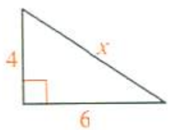

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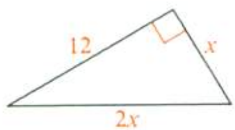

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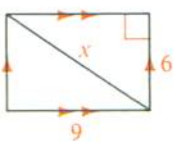

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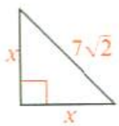

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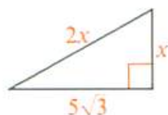
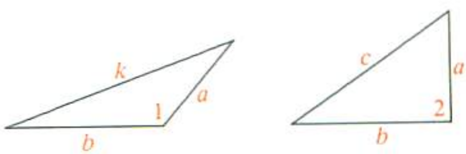


19. In the figure, $m\angle 1 > 90$ and $m\angle 2 = 90$. Two sides of the obtuse triangle are congruent to two sides of the right triangle as shown. State a reason to support each statement.
   a. $k > c$
   b. $k^2 > c^2$
   c. $c^2 = a^2 + b^2$
   d. $k^2 > a^2 + b^2$



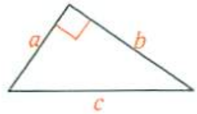
20. Exercise 19 outlines a proof. Complete the statement of what is proved: In an obtuse triangle, the square of the side opposite the obtuse angle is $\underline{\ ?\ }$.

Written Exercises

Copy and complete the table.

A

	1.	2.	3.	4.	5.	6.	7.	8.
a	3	?	5	?	11	$3n$	?	6
b	4	8	?	15	?	$4n$	$3\sqrt{2}$	$6\sqrt{3}$
c	?	10	13	17	61	?	6	?



Draw a figure for each exercise. The length and width of a rectangle are given. Find the length of a diagonal.

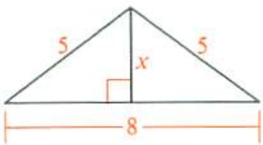
9. 8 and 6 **10.** 0.8 and 0.6 **11.** 80 and 60 **12.** $\sqrt{3}$ and 1

The length of a diagonal of a square is given. Find the length of a side of the square.

13. 2 **14.** 10 **15.** $20k$ **16.** $7n\sqrt{2}$

Find the value of x in each figure.

B **17.**



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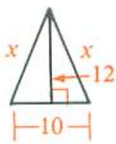

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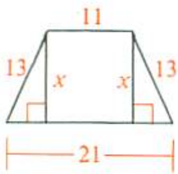

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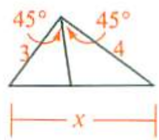

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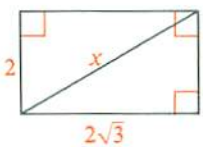

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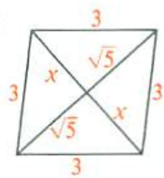

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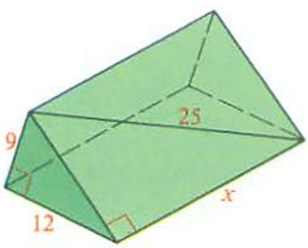

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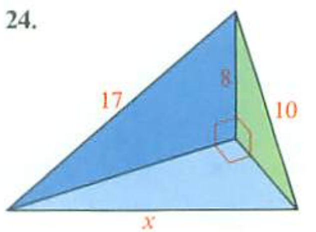
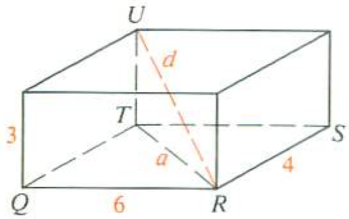
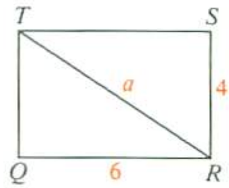


In Exercises 25-30, the dimensions of a rectangular box are given. Find the length of a diagonal of the solid.

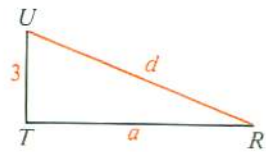
Example Dimensions 6, 4, 3

Solution





$a^2 = 6^2 + 4^2$
$a^2 = 36 + 16$
$a^2 = 52$



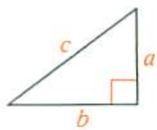
$d^2 = a^2 + 3^2$
$d^2 = 52 + 9$
$d^2 = 61$
$d = \sqrt{61}$

25. 12, 4, 3 **26.** 5, 5, 2 **27.** $\sqrt{7}$, $\sqrt{6}$, $\sqrt{5}$

28. e, e, e **29.** l, w, h **30.** $n + 2$, $\sqrt{2n + 1}$, 2

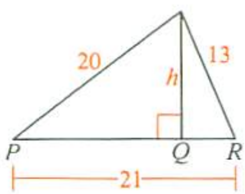
31. See Classroom Exercise 20. State and prove a theorem about a side opposite an acute angle in any triangle.

C 32. Given: A rt. △ with sides a, b, c;
b is the arithmetic mean of a and c.
Prove: $a:b:c = 3:4:5$



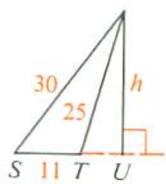
Find the value of *h*.

33.



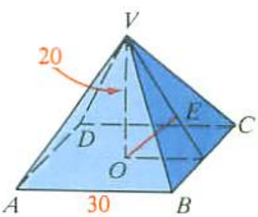
(*Hint:* Let $PQ = x$; $QR = 21 - x$. Use two right triangles.)

34.



(*Hint:* Let $TU = x$; $SU = x + 11$.)

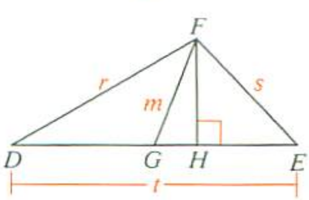
35. O is the *center* of square $ABCD$ (the point of intersection of the diagonals) and $\overline{VO}$ is perpendicular to the plane of the square. Find OE, the distance from O to the plane of $\triangle VBC$.



36. Given: $\triangle DEF$ with median $\overline{FG}$

Prove: $m = \frac{1}{2}\sqrt{2r^2 + 2s^2 - t^2}$

(*Hint:* Draw the perpendicular from F to $\overline{DE}$. Let $GH = x$ and $FH = y$.)



Self-Test 1

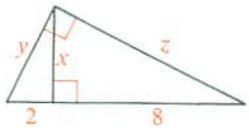
1. Simplify the expressions.

   a. $\sqrt{48}$ b. $\frac{7}{\sqrt{2}}$ c. $\sqrt{\frac{5}{8}}$ d. $\frac{3\sqrt{20}}{4}$

2. Find the geometric mean between 3 and 15.

The diagram shows the altitude drawn to the hypotenuse of a right triangle.

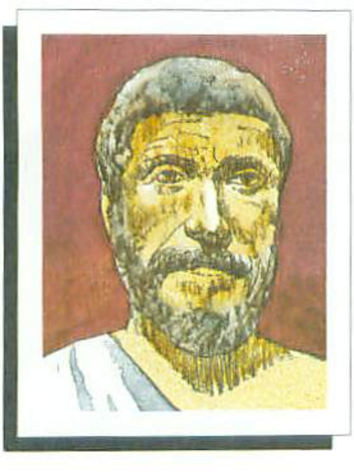
3. $x =$ __?__
4. $y =$ __?__
5. $z =$ __?__



The hypotenuse of a right triangle is h. The legs are f and g. Copy and complete the table.

	6.	7.	8.	9.	10.
f	6	?	?	?	n
g	8	12	$5\sqrt{2}$	$7\sqrt{3}$	n
h	?	13	10	14	?

B I O G R A P H I C A L N O T E



Pythagoras

Most of the ancient Greeks believed the whole numbers were sufficient to represent all the physical universe. Pythagoras (sixth century B.C.) came to realize that whole numbers and ratios of whole numbers were an inadequate representation of the physical world.

He discovered that the diagonal of a square one unit on each side has a length that can never be represented as the ratio of two integers. Although his best-known work is the Pythagorean Theorem, he also made numerous other contributions to mathematics. Among these is the proof that the sum of a triangle's three angles is equal to two right angles. Pythagoras also discovered the dependence of musical intervals on arithmetical ratios.

Right Triangles

Objectives

1. State and apply the converse of the Pythagorean Theorem and related theorems about obtuse and acute triangles.
2. Determine the lengths of two sides of a 45°-45°-90° or a 30°-60°-90° triangle when the length of the third side is known.

6-3 The Converse of the Pythagorean Theorem

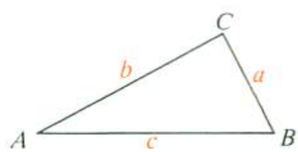
We have seen that the converse of a theorem is not necessarily true. However, the converse of the Pythagorean Theorem *is* true. It is stated below as our next theorem.

Theorem 6-3

If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

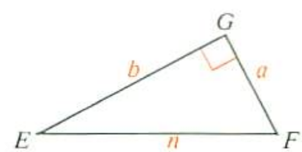
Given: $\triangle ABC$ with $c^2 = a^2 + b^2$

Prove: $\triangle ABC$ is a rt. $\triangle$.



Outline of proof:

1. Draw rt. $\triangle EFG$ with legs a and b.
2. $n^2 = a^2 + b^2$ (Why?)
3. $c^2 = a^2 + b^2$ (Given)
4. $c = n$ (Why?)
5. $\triangle ABC \cong \triangle EFG$ (SSS Postulate)
6. $\angle C$ is a rt. $\angle$. (Why?)
7. $\triangle ABC$ is a rt. $\triangle$. (Why?)



A triangle with sides 3 units, 4 units, and 5 units long is called a 3-4-5 triangle. The numbers 3, 4, and 5 satisfy the equation $a^2 + b^2 = c^2$, so we can apply Theorem 6-3 to conclude that a 3-4-5 triangle is a right triangle. Furthermore, any triangle similar to a 3-4-5 triangle must be a right triangle. The side lengths shown in the table at the top of the next page all satisfy the equation

$$a^2 + b^2 = c^2,$$

so the triangles formed are right triangles.

Some Common Right Triangle Lengths

3, 4, 5	5, 12, 13	8, 15, 17	7, 24, 25
6, 8, 10	10, 24, 26		
9, 12, 15			
12, 16, 20			
15, 20, 25			

When a triangle is not a right triangle, the squares of the sides can be used to determine whether it is obtuse or acute, as shown in Theorems 6-4 and 6-5.

Theorem 6-4

If the square of the longest side of a triangle is greater than the sum of the squares of the other two sides, then the triangle is an obtuse triangle.

Theorem 6-5

If the square of the longest side of a triangle is less than the sum of the squares of the other two sides, then the triangle is an acute triangle.

Example Determine whether a triangle formed with sides having the lengths named is acute, right, or obtuse.

a. 9, 40, 41 **b.** 6, 7, 8

Solution

a. $9^2 + 40^2 \; \underline{\ ?\ } \; 41^2$

$81 + 1600 \; \underline{\ ?\ } \; 1681$

$1681 = 1681$

The triangle is right.

b. $6^2 + 7^2 \; \underline{\ ?\ } \; 8^2$

$36 + 49 \; \underline{\ ?\ } \; 64$

$85 > 64$

The triangle is acute.

Classroom Exercises

If a triangle is formed with sides having the lengths named, is it acute, right, or obtuse? If a triangle can't be formed, say *not possible*.

1. 1, 4, 6 **2.** 4, 6, 8 **3.** 6, 8, 10

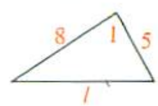
4. 8, 10, 12 **5.** $\sqrt{7}$, $\sqrt{7}$, $\sqrt{14}$ **6.** 4, $4\sqrt{3}$, 8

7. Specify all lengths l that will make the statement true.

a. $\angle 1$ is a right $\angle$. **b.** $\angle 1$ is an acute $\angle$.

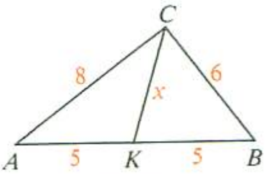
c. $\angle 1$ is an obtuse $\angle$. **d.** The triangle is isosceles.

e. No triangle is possible.

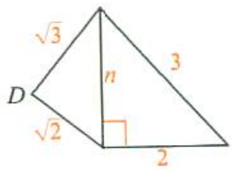


For Exercises 8–10, refer to the figures below.

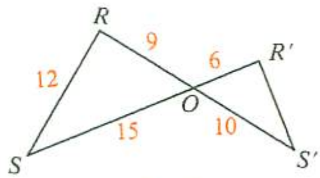
8. Explain why x must equal 5.
9. Explain why $\angle D$ must be a right angle.
10. Explain why $\angle R'$ must be a right angle.



Ex. 8



Ex. 9



Ex. 10

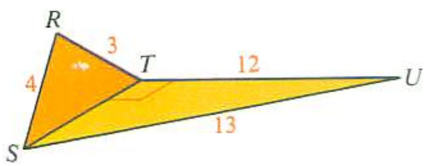
Written Exercises

Tell whether a triangle formed with sides having the lengths named is acute, right, or obtuse. When it isn't possible to decide, write *can't tell.*

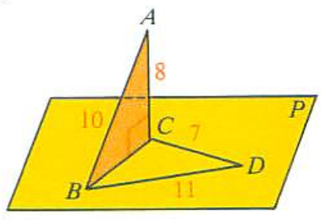
A

1. 11, 11, 15
2. 9, 9, 13
3. 8, $8\sqrt{3}$, 16
4. 0.03, 0.04, 0.05
5. 300, 400, 501
6. 0.6, 0.8, 1
7. $5n$, $12n$, $13n$ where $n > 0$
8. $n + 4$, $n + 5$, $n + 6$ where $n \geq 1$
9. $7 - n$, 7, $7 + n$ where $0 < n \leq 3$

10. Given: $\angle UTS$ is a rt. $\angle$.
    Explain why $\triangle TRS$ must be a right triangle.



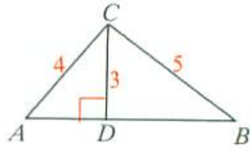
11. Given: $\overline{AC} \perp$ plane P
    Explain why $\triangle BCD$ must be an obtuse triangle.



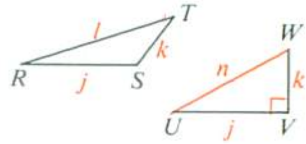
B

12. Sketch $\square ABCD$ with $AB = 13$, $AC = 24$, and $BD = 10$. What special kind of parallelogram is $ABCD$? Explain your answer.
13. Sketch $\square RSTU$, with diagonals intersecting at M. $RS = 9$, $ST = 20$, and $RM = 11$. Which segment is longer, $\overline{SM}$ or $\overline{RM}$? Explain your answer.
14. Given: A triangle with sides a, b, c;
    $a = n^2 - 1$; $b = 2n$; $c = n^2 + 1$
    Prove: The triangle is a rt. $\triangle$.
15. The sides of a triangle have lengths x, $x + 4$, and 20. Specify those values of x for which the triangle is acute with longest side 20.

16. Is $\triangle ABC$ acute, right, or obtuse? Explain your answer.

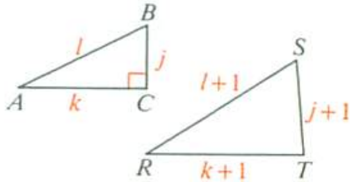


C 17. Write a plan for the proof of Theorem 6-4.
Given: $\triangle RST$; $l^2 > j^2 + k^2$
Prove: $\triangle RST$ is an obtuse triangle.
(*Hint*: Start by drawing right $\triangle UVW$ with legs j and k. Compare lengths l and n.)

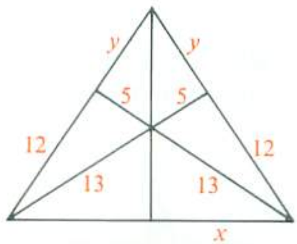


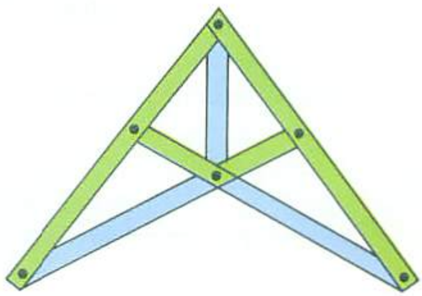
18. Write a plan for the proof of Theorem 6-5.

19. Given: $\triangle ABC$ and $\triangle RST$, with sides having the lengths shown; $\angle C$ is a rt. $\angle$.
Prove: $\triangle RST$ is an acute $\triangle$.



20. Find the values of x and y. *Note:* A frame in this shape, like the simple *scissors truss* shown at the right below, can be used to support a peaked roof. The weight of the roof compresses some parts of the frame (green), while other parts are in tension (blue). A frame made with s segments joined at j points is stable if $s \geq 2j - 3$. In the truss shown, 9 segments connect 6 points. Verify that the truss is stable.





COMPUTER KEY-IN

Suppose a, b, and c are positive integers such that $a^2 + b^2 = c^2$. Then the converse of the Pythagorean Theorem guarantees that a, b, and c are the lengths of the sides of a right triangle. Because of this, any such triple of integers is called a **Pythagorean triple.**

For example, 3, 4, 5 is a Pythagorean triple since $3^2 + 4^2 = 5^2$. Another triple is 6, 8, 10, since $6^2 + 8^2 = 10^2$. The triple 3, 4, 5 is called a *primitive* Pythagorean triple because no factor (other than 1) is common to all three integers. 6, 8, 10 is *not* a primitive triple.

The following program in BASIC lists some Pythagorean triples.

```
10 FOR X = 2 TO 7
20 FOR Y = 1 TO X - 1
30 LET A = 2 * X * Y
40 LET B = X * X - Y * Y
50 LET C = X * X + Y * Y
60 PRINT A;",";B;",";C
70 NEXT Y
80 NEXT X
90 END
```

Exercises

1. Type and RUN the program. (If your computer uses a language other than BASIC, write and RUN a similar program.) What Pythagorean triples did it list? Which of these are primitive Pythagorean triples?

2. The program above uses a method for finding Pythagorean triples that was developed by Euclid around 320 B.C. His method can be stated as follows:

If x and y are positive integers with $y < x$, then $a = 2xy$, $b = x^2 - y^2$, and $c = x^2 + y^2$ is a Pythagorean triple.

To verify that Euclid's method is correct, show that the equation below is true.

$$(2xy)^2 + (x^2 - y^2)^2 = (x^2 + y^2)^2$$

3. Explain why, in line 20, the last value for Y must be X − 1. What happens if Y = X?

4. To find several Pythagorean triples containing large numbers change lines 10 and 20 to

```
10 FOR X = 51 TO 55
20 FOR Y = 50 TO X - 1
```

and RUN the program.

5. In Exercise 1 the computer printed 21 Pythagorean triples. Suppose the values of X are the integers from 2 to 10 and the values of Y are the integers from 1 to X − 1. How many Pythagorean triples would you expect the computer to print? RUN the program to verify your answer.

6. Suppose the values of X are the integers from 2 to 100 and the values of Y are the integers from 1 to X − 1. How many Pythagorean triples should the computer print?

Challenge

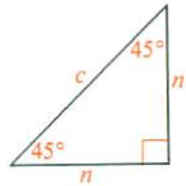
A room is 30 ft long, 12 ft wide, and 12 ft high. A spider is at the middle of an end wall, 1 ft from the floor. A fly is at the middle of the other end wall, 1 ft from the ceiling, too frightened to move. The spider crawls to the fly. What is the shortest distance?

6-4 Special Right Triangles

An isosceles right triangle is also called a 45°-45°-90° triangle, because the measures of the angles are 45, 45, and 90.

Theorem 6-6 45°-45°-90° Theorem

In a 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times as long as a leg.



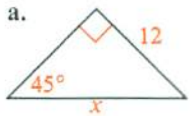
Given: A 45°-45°-90° triangle

Prove: hypotenuse = $\sqrt{2} \cdot$ leg

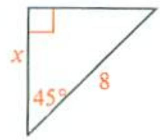
Plan for Proof: Let the sides of the given triangle be n, n, and c. Apply the Pythagorean Theorem and solve for c in terms of n.

Example 1 Find the value of x.

a.



b.



Solution

a. hyp. $= \sqrt{2} \cdot$ leg

$x = \sqrt{2} \cdot 12$

$x = 12\sqrt{2}$

b. hyp. $= \sqrt{2} \cdot$ leg

$8 = \sqrt{2} \cdot x$

$$x = \frac{8}{\sqrt{2}} = \frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{8\sqrt{2}}{2}$$

$x = 4\sqrt{2}$

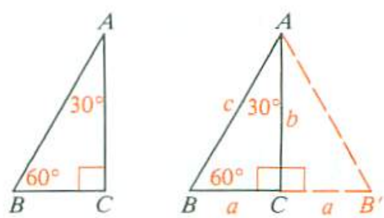
Another special right triangle has acute angles measuring 30 and 60.

Theorem 6-7 30°-60°-90° Theorem

In a 30°-60°-90° triangle, the hypotenuse is twice as long as the shorter leg, and the longer leg is $\sqrt{3}$ times as long as the shorter leg.

Given: $\triangle ABC$, a 30°-60°-90° triangle

Prove: hypotenuse = $2 \cdot$ shorter leg
longer leg = $\sqrt{3} \cdot$ shorter leg

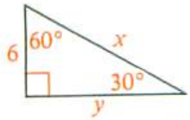


Outline of proof:

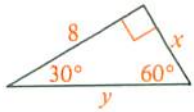
1. Build onto $\triangle ABC$ as shown.
2. $\triangle AB'C \cong \triangle ABC$, so $m\angle B' = 60$
3. $\triangle ABB'$ is equiangular. (Why?)
4. $\triangle ABB'$ is equilateral, so $c = 2a$.
5. $a^2 + b^2 = c^2$ (Why?)
6. $a^2 + b^2 = 4a^2$ (Why?)
7. $b^2 = 3a^2$ and $b = a\sqrt{3}$ (Why?)

Example 2 Find the values of x and y.

a.



b.



Solution

a. hyp. = 2 · shorter leg

$x = 2 \cdot 6$

$x = 12$

longer leg = $\sqrt{3}$ · shorter leg

$y = 6\sqrt{3}$

b. longer leg = $\sqrt{3}$ · shorter leg

$8 = \sqrt{3}x$

$x = \dfrac{8}{\sqrt{3}} = \dfrac{8\sqrt{3}}{3}$

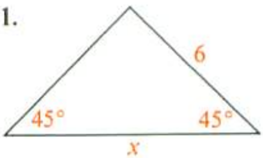
hyp. = 2 · shorter leg

$y = 2 \cdot \dfrac{8\sqrt{3}}{3} = \dfrac{16\sqrt{3}}{3}$

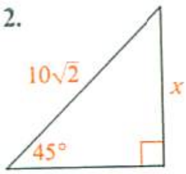
Classroom Exercises

Find the value of x.

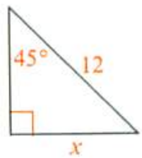
1.



2.

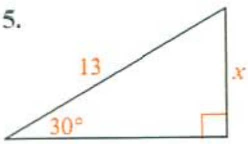


3.

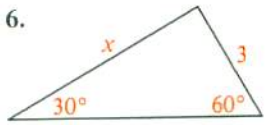


4.

5.



6.



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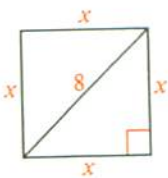

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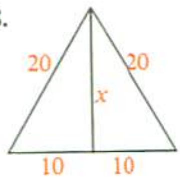

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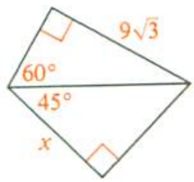
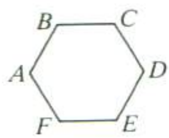
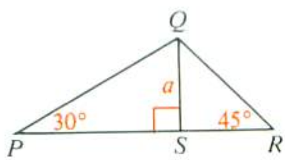


10. In regular hexagon $ABCDEF$, $AB = 8$. Find AC and AD.



11. Express PQ, PS, and QR in terms of a.



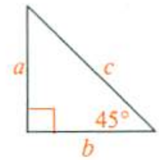
Written Exercises

A 1. Draw an equilateral triangle with sides 10 units long. Draw an altitude. On your diagram, show the length of each segment of the base.
a. Use Theorem 6-7 to find the length of the altitude.
b. Use the Pythagorean Theorem to find the length of the altitude.

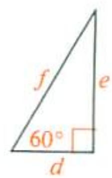
2. Draw a square with a diagonal 13 units long.
a. Use Theorem 6-6 to find the length of a side of the square.
b. Use the Pythagorean Theorem to find the length of a side of the square.

Copy and complete the tables.

	3.	4.	5.	6.	7.	8.
a	6	?	$\sqrt{5}$	?	?	?
b	?	$\frac{4}{5}$	?	?	?	?
c	?	?	?	$8\sqrt{2}$	6	$\sqrt{22}$

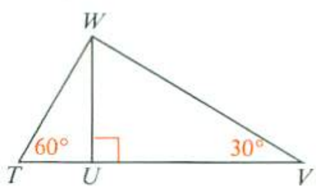


	9.	10.	11.	12.	13.	14.
d	7	$\frac{1}{5}$	?	?	?	?
e	?	?	$5\sqrt{3}$	6	?	?
f	?	?	?	?	12	5

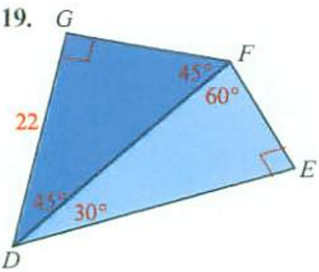


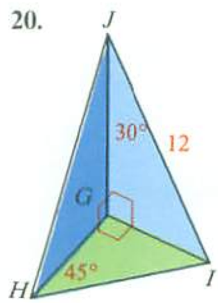
Copy and complete the table. Draw a new sketch for each exercise and label lengths as you find them.

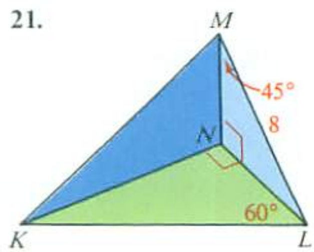
	TU	UV	TV	WT	WU	WV
B 15.	7	?	?	?	?	?
16.	?	?	$8\sqrt{3}$	?	?	?
17.	?	?	?	50	?	?
18.	?	?	?	?	7	?



Find the lengths of as many segments as possible.

19. 

20. 

21. 

22. The diagonals of a rectangle are 8 units long and intersect at a $60°$ angle. Find the dimensions of the rectangle.

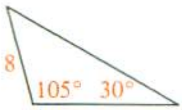
23. The sum of the lengths of the sides of a rhombus is 64 and one of its angles has measure 120. Find the lengths of the diagonals.

24. Prove Theorem 6-6.

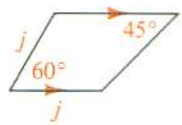
25. Explain why any triangle having sides in the ratio $1:\sqrt{3}:2$ must be a $30°$-$60°$-$90°$ triangle.

C 26. In quadrilateral $QRST$: $m\angle R = 60$; $m\angle T = 90$; $QR = RS$; $ST = 8$; $TQ = 8$. How long is the longer diagonal of the quadrilateral?

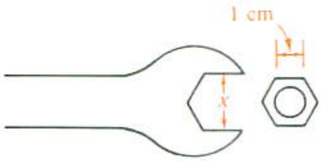
27. Find the perimeter of the triangle.



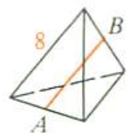
28. Find the length of the median of the trapezoid in terms of j.



29. If the wrench just fits the hexagonal nut, what is the value of x?



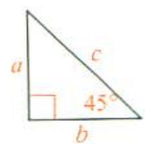
★ 30. The six edges of the solid shown are 8 units long. A and B are midpoints of two edges as shown. Find AB.



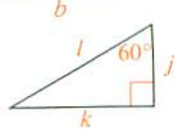
Self-Test 2

Three sides of a triangle are given. Tell whether the triangle is acute, right, or obtuse.

1. 11, 60, 61
2. 7, 9, 11
3. 0.2, 0.3, 0.4



4. If $a = 5$, then $b =$ _?_ and $c =$ _?_.
5. If $c = 12$, then $a =$ _?_ and $b =$ _?_.



6. If $j = 4$, then $k =$ _?_ and $l =$ _?_.
7. If $l = 10$, then $j =$ _?_ and $k =$ _?_.
8. If $k = 6$, then $j =$ _?_ and $l =$ _?_.
9. The sides of a rhombus are 4 units long, and one diagonal has length 4. How long is the other diagonal?

Trigonometry

Objectives

1. Define the tangent, sine, and cosine ratios for an acute angle.
2. Solve right triangle problems by correct selection and use of the tangent, sine, and cosine ratios.

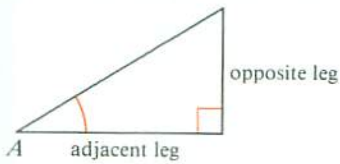
6-5 The Tangent Ratio

The word *trigonometry* comes from Greek words that mean "triangle measurement." Our study in this book will be limited to the trigonometry of right triangles. In the right triangle shown, one acute angle is marked. The leg opposite this angle and the leg adjacent to this angle are labeled.

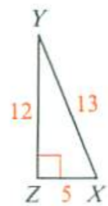
The following ratio of the lengths of the legs is called the *tangent ratio*.

tangent of $\angle A = \dfrac{\text{leg opposite } \angle A}{\text{leg adjacent to } \angle A}$

In abbreviated form: $\tan A = \dfrac{\text{opposite}}{\text{adjacent}}$



Example 1 Find tan X and tan Y.



Solution

$$\tan X = \frac{\text{leg opposite } \angle X}{\text{leg adjacent to } \angle X} = \frac{12}{5}$$

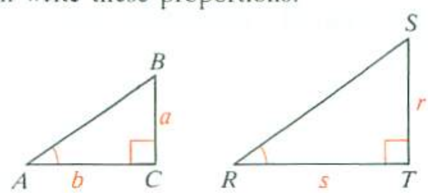
$$\tan Y = \frac{\text{leg opposite } \angle Y}{\text{leg adjacent to } \angle Y} = \frac{5}{12}$$

In the right triangles shown below, $m\angle A = m\angle R$. Then by the AA Similarity Postulate, the triangles are similar. We can write these proportions:

$\dfrac{a}{r} = \dfrac{b}{s}$ (Why?)

$\dfrac{a}{b} = \dfrac{r}{s}$ (A property of proportions)

$\tan A = \tan R$ (Def. of tangent ratio)



We have shown that if $m\angle A = m\angle R$, then $\tan A = \tan R$. Thus we have shown that the value of the tangent of an angle depends only on the size of the angle, not on the size of the right triangle. It is also true that if $\tan A = \tan R$ for acute angles A and R, then $m\angle A = m\angle R$.

Since the tangent of an angle depends only on the measure of the angle, we can write $\tan 10°$, for example, to stand for the tangent of any angle with a degree measure of 10. The table on page 271 lists the values of the tangents of some angles with measures between 0 and 90. Most of the values are approximations, rounded to four decimal places. Suppose you want the approximate value of tan 33°. Locate 33° in the angle column. Go across to the tangent column. Read .6494. You write $\tan 33° \approx 0.6494$, where the symbol $\approx$ means "is approximately equal to."

Example 2 Find the value of y to the nearest tenth.

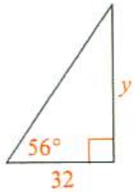
Solution

$$\tan 56° = \frac{y}{32}$$

$$1.4826 \approx \frac{y}{32}$$

$$32(1.4826) \approx y$$

$$y \approx 47.4432, \text{ or } 47.4$$



You can find the degree measure of an angle with a given tangent by reading from the tangent column across to the angle column.

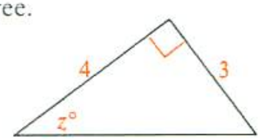
Example 3 Find the value of $z°$ to the nearest degree.

Solution

$$\tan z° = \frac{3}{4}$$

$$\tan z° = 0.7500$$

$$z° \approx 37°$$



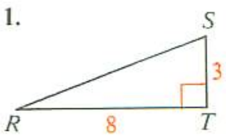
Notice that 0.7500 falls between two values in the tangent column: $\tan 36° \approx 0.7265$ and $\tan 37° \approx 0.7536$. Since 0.7500 is closer to 0.7536, we use 37° as an approximate value for $z°$.

A scientific calculator with keys for computing trigonometric ratios can be used to find the tangent of a given angle. If the calculator has an inverse key the measure of the angle with a given tangent can also be found. Like values found from a table, the values given by a calculator are approximations, usually rounded to seven or eight decimal places.

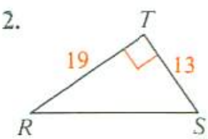
Classroom Exercises

In Exercises 1-3 express tan R as a ratio.

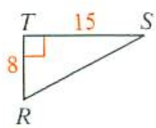
1.



2.



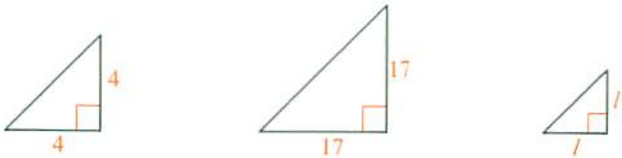
3.



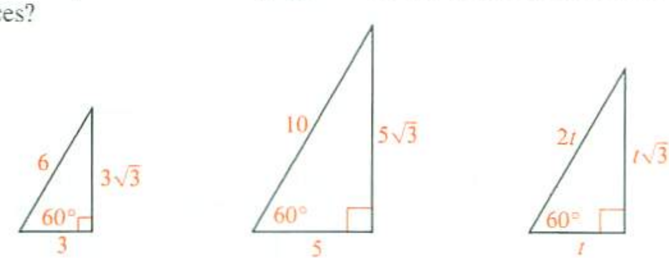
4-6. Express tan S as a ratio for each triangle above.

7. Use the table on page 271 to complete the statements.
   a. $\tan 24° \approx$ __?__ b. $\tan 41° \approx$ __?__ c. $\tan 88° \approx$ __?__
   d. tan __?__ ≈ 2.4751 e. tan __?__ ≈ 0.3057 f. tan __?__ ≈ 0.8098

8. Three 45°-45°-90° triangles are shown below.
   a. In each triangle, express tan 45° in simplified form.
   b. See the entry for tan 45° on page 271. Is the entry exact?



9. Three 30°-60°-90° triangles are shown below.
   a. In each triangle, express tan 60° in simplified radical form.
   b. Because $\tan 60° = \frac{\sqrt{3}}{1}$, and because $\sqrt{3} \approx 1.732051$, you can write $\tan 60° \approx$ __?__.
   c. Is the entry for tan 60° on page 271 exact? Is it correct to four decimal places?



10. Notice that the tangent values increase rapidly toward the end of the table on page 271. Explain how you know that there is some angle with a tangent value equal to 1,000,000. Is there any upper limit to tangent values?

11. Two ways to find the value of x are started below.

Using tan 40°:

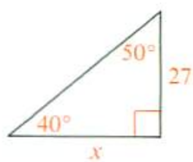
$\tan 40° = \frac{27}{x}$

$0.8391 \approx \frac{27}{x}$

Using tan 50°:

$\tan 50° = \frac{x}{27}$

$1.1918 \approx \frac{x}{27}$



Which of the following statements are correct?

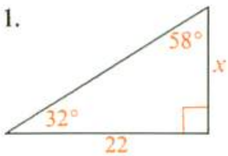
a. $x \approx 27 \cdot 0.8391$ b. $x \approx \frac{27}{0.8391}$

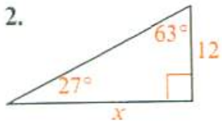
c. $x \approx 27 \cdot 1.1918$ d. $x \approx \frac{27}{1.1918}$

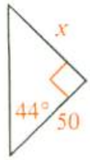
Which method is better if you are not using a calculator for the arithmetic?

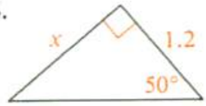
Written Exercises

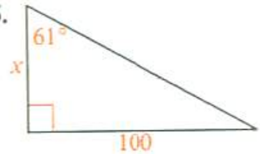
Find x correct to the nearest tenth. Use the table on page 271.

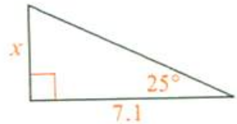
A 1.


2.


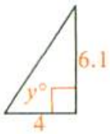
3.


4.


5.


6.


Find $y°$ correct to the nearest degree.

7.


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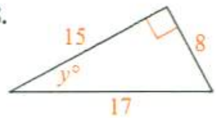

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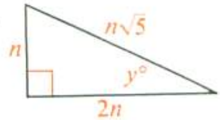
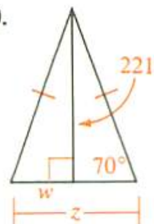
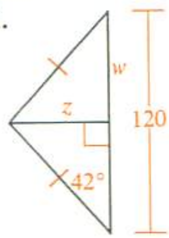
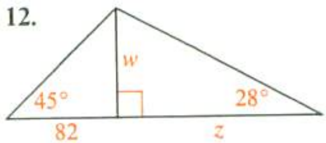


Find w, then z, correct to the nearest integer.

B 10.


11.


12.


13. A rhombus has diagonals of length 4 and 10. Find the angles of the rhombus to the nearest degree.

14. The sides of a rectangle are 20 and 40. Find, to the nearest degree, the measure of an acute angle formed at the intersection of the diagonals.

15. A natural question to consider is the following:

$$\text{Does } \tan A + \tan B = \tan(A + B)?$$

Try 35 for the measure of A and 25 for the measure of B.

a. $\tan 35° + \tan 25° \approx$ _?_ + _?_ = _?_

b. $\tan(35° + 25°) = \tan$ _?_° $\approx$ _?_

c. What is your answer to the general question raised in this exercise, *yes* or *no*?

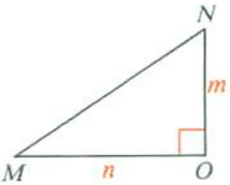
16. The shorter diagonal of a rhombus with a 70° angle is 124 cm long. How long (to the nearest centimeter) is the longer diagonal?

17. Complete the proof by supplying reasons and completing statements.

Given: $\angle M$ and $\angle R$ are complementary angles.

Prove: $\tan M \cdot \tan R = 1$

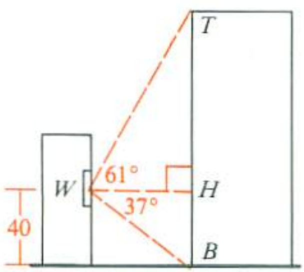
Plan for Proof: Draw right $\triangle MNO$ with $\angle M$ at one vertex and a right angle at O. The other acute angle, $\angle N$, is complementary to $\angle M$, so $\angle N \cong \angle R$. Show that $\tan M \cdot \tan N = 1$ and conclude that $\tan M \cdot \tan R = 1$.



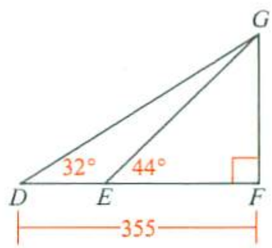
Proof:

1. $\angle N$ is a complement of $\angle M$. (Why?)
2. $\angle R$ is a complement of $\angle M$. (Why?)
3. $m\angle N = m\angle R$ (Why?)
4. $\tan N = \tan R$ (Why?)
5. $\tan M = \frac{m}{n}$ and $\tan N = \underline{\ ?\ }$ (Definition of tangent)
6. $\tan M \cdot \tan N = \frac{m}{n} \cdot \frac{n}{m} = 1$ (Multiplication and Substitution Properties)
7. $\tan M \cdot \tan R = 1$ (Why?)

C 18. A person at window W, 40 ft above street level, sights points on a building directly across the street. H is chosen so that $\overline{WH}$ is horizontal. T is directly above H, and B is directly below. By measurement, $m\angle TWH = 61$ and $m\angle BWH = 37$. How far above street level is T?



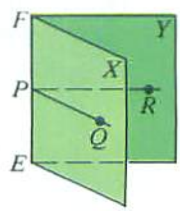
Ex. 18



Ex. 19

19. Use the figure to find EF to the nearest integer.

20. In the diagram, half-planes X and Y with the same edge $\overleftrightarrow{FE}$ form **dihedral angle** X-FE-Y. If $\overrightarrow{PR} \perp \overleftrightarrow{FE}$ and $\overrightarrow{PQ} \perp \overleftrightarrow{FE}$, then $\angle RPQ$ is called a *plane angle* of this dihedral angle. The measure of a dihedral angle is defined to be the measure of any of its plane angles, so the measure of this dihedral angle is $m\angle RPQ$.



In the cube shown at the right, find, to the nearest degree, the measure of the dihedral angle containing the two shaded triangles.

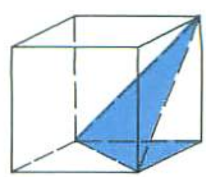


Table of Trigonometric Ratios

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	**46°**	.7193	.6947	1.0355
2°	.0349	.9994	.0349	**47°**	.7314	.6820	1.0724
3°	.0523	.9986	.0524	**48°**	.7431	.6691	1.1106
4°	.0698	.9976	.0699	**49°**	.7547	.6561	1.1504
5°	.0872	.9962	.0875	**50°**	.7660	.6428	1.1918
6°	.1045	.9945	.1051	**51°**	.7771	.6293	1.2349
7°	.1219	.9925	.1228	**52°**	.7880	.6157	1.2799
8°	.1392	.9903	.1405	**53°**	.7986	.6018	1.3270
9°	.1564	.9877	.1584	**54°**	.8090	.5878	1.3764
10°	.1736	.9848	.1763	**55°**	.8192	.5736	1.4281
11°	.1908	.9816	.1944	**56°**	.8290	.5592	1.4826
12°	.2079	.9781	.2126	**57°**	.8387	.5446	1.5399
13°	.2250	.9744	.2309	**58°**	.8480	.5299	1.6003
14°	.2419	.9703	.2493	**59°**	.8572	.5150	1.6643
15°	.2588	.9659	.2679	**60°**	.8660	.5000	1.7321
16°	.2756	.9613	.2867	**61°**	.8746	.4848	1.8040
17°	.2924	.9563	.3057	**62°**	.8829	.4695	1.8807
18°	.3090	.9511	.3249	**63°**	.8910	.4540	1.9626
19°	.3256	.9455	.3443	**64°**	.8988	.4384	2.0503
20°	.3420	.9397	.3640	**65°**	.9063	.4226	2.1445
21°	.3584	.9336	.3839	**66°**	.9135	.4067	2.2460
22°	.3746	.9272	.4040	**67°**	.9205	.3907	2.3559
23°	.3907	.9205	.4245	**68°**	.9272	.3746	2.4751
24°	.4067	.9135	.4452	**69°**	.9336	.3584	2.6051
25°	.4226	.9063	.4663	**70°**	.9397	.3420	2.7475
26°	.4384	.8988	.4877	**71°**	.9455	.3256	2.9042
27°	.4540	.8910	.5095	**72°**	.9511	.3090	3.0777
28°	.4695	.8829	.5317	**73°**	.9563	.2924	3.2709
29°	.4848	.8746	.5543	**74°**	.9613	.2756	3.4874
30°	.5000	.8660	.5774	**75°**	.9659	.2588	3.7321
31°	.5150	.8572	.6009	**76°**	.9703	.2419	4.0108
32°	.5299	.8480	.6249	**77°**	.9744	.2250	4.3315
33°	.5446	.8387	.6494	**78°**	.9781	.2079	4.7046
34°	.5592	.8290	.6745	**79°**	.9816	.1908	5.1446
35°	.5736	.8192	.7002	**80°**	.9848	.1736	5.6713
36°	.5878	.8090	.7265	**81°**	.9877	.1564	6.3138
37°	.6018	.7986	.7536	**82°**	.9903	.1392	7.1154
38°	.6157	.7880	.7813	**83°**	.9925	.1219	8.1443
39°	.6293	.7771	.8098	**84°**	.9945	.1045	9.5144
40°	.6428	.7660	.8391	**85°**	.9962	.0872	11.4301
41°	.6561	.7547	.8693	**86°**	.9976	.0698	14.3007
42°	.6691	.7431	.9004	**87°**	.9986	.0523	19.0811
43°	.6820	.7314	.9325	**88°**	.9994	.0349	28.6363
44°	.6947	.7193	.9657	**89°**	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

General Contractor

Every construction project, from remodeling a kitchen to building a skyscraper, requires someone with overall responsibility for getting the job done efficiently and on time. Once the planning phases of a project are completed, the general contractor takes charge.

Construction involves many specialized kinds of work, all of them interrelated. For example, walls cannot be finished until electrical wiring has been completed. The contractor must arrange for the proper personnel to do the work, schedule the sequence of operations carefully, and monitor the pace and quality of work continuously. Effective

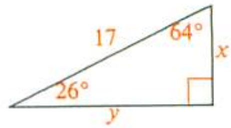
management of the project will avoid delays and increases in the cost of the structure. Coordination, scheduling, and supervision are key aspects of the contractor's work. On small projects, the contractor may also do a significant part of the actual construction work in addition to managing the whole project.

Most contractors begin their careers by first mastering one or more trades, such as carpentry or masonry. After acquiring enough experience

and expertise, a person may decide to go into business as a general contractor. Currently, some colleges are beginning to offer degrees in construction, providing another path for becoming a contractor.

6-6 The Sine and Cosine Ratios

Can you find the values of x and y in the diagram by using tan 64°? by using tan 26°? The answer is *no,* because the only side known, the hypotenuse, is not involved in the definition of tangent. Two ratios that do relate the hypotenuse to the legs are the *sine* and *cosine.*



$$\textbf{sine} \text{ of } \angle A = \frac{\text{leg opposite } \angle A}{\text{hypotenuse}}$$

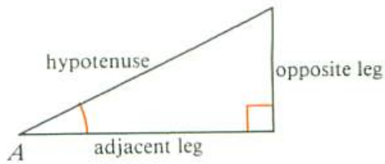
$$\textbf{cosine} \text{ of } \angle A = \frac{\text{leg adjacent to } \angle A}{\text{hypotenuse}}$$

We now have three useful trigonometric ratios:

$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$



Example 1 Find the values of x and y correct to the nearest integer. Use the table on page 271.

Solution

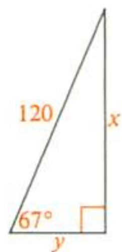
$$\sin 67^\circ = \frac{x}{120} \qquad \cos 67^\circ = \frac{y}{120}$$

$$0.9205 \approx \frac{x}{120} \qquad 0.3907 \approx \frac{y}{120}$$

$$120 \cdot 0.9205 \approx x \qquad 120 \cdot 0.3907 \approx y$$

$$110.46 \approx x \qquad 46.884 \approx y$$

$$x \approx 110 \qquad y \approx 47$$



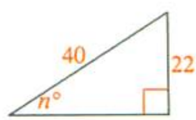
Example 2 Find $n°$ correct to the nearest degree.

Solution

$$\sin n^\circ = \frac{22}{40}$$

$$\sin n^\circ = 0.5500$$

$$n^\circ \approx 33^\circ$$

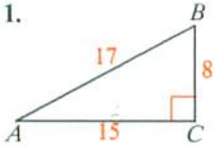


(The two values in the sine column closest to 0.5500 are 0.5446 and 0.5592. The closer is 0.5446.)

Classroom Exercises

In Exercises 1–3, express sin *A*, cos *A*, and tan *A* as fractions.

1.



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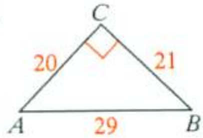
3.



4–6. Using the triangles in Exercises 1–3, express sin *B*, cos *B*, and tan *B* as fractions.

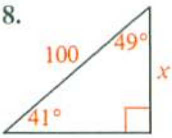
7. Use the table on page 271 to complete the statements.

a. $\sin 13° \approx$ __?__ **b.** $\cos 88° \approx$ __?__

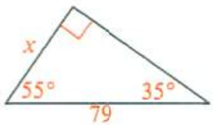
c. sin __?__ ≈ 0.7547 **d.** cos __?__ ≈ 0.9511

State two different equations you could use to find the value of *x*.

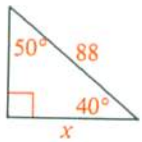
8.



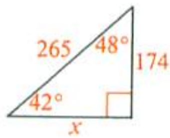
9.



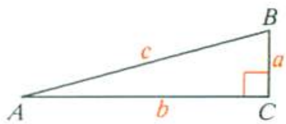
10.



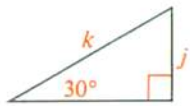
11. State four different equations you could use to find the value of *x*.



12. The word cosine is related to the phrase complement's sine. Explain the relationship by using the diagram to express the cosine of $\angle A$ and the sine of its complement, $\angle B$.

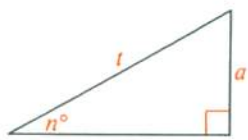


13. The table on page 271 lists 0.5000 as the value of sin 30°. This value is exact. Explain why.



14. Suppose $\sin n° = \frac{15}{17}$. Find $\cos n°$ without using a table.

15. Suppose $\sin n° = \frac{a}{t}$. Express $\cos n°$ algebraically.



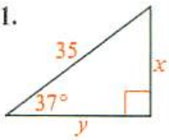
16. According to the table on page 271, sin 1° and tan 1° are both approximately 0.0175. Which is actually larger? How do you know?

Written Exercises

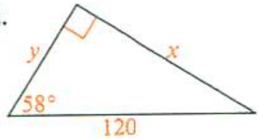
In these exercises, use the table on page 271. Find lengths correct to the nearest integer and angle measures to the nearest degree.

In Exercises 1–9, find the values of the variables.

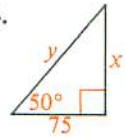
A **1.**



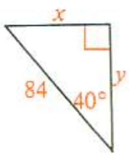
2.



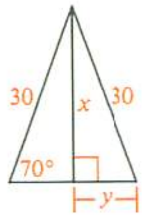
3.



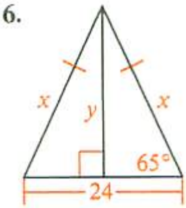
4.



5.



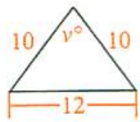
6.



7.

8.

9.

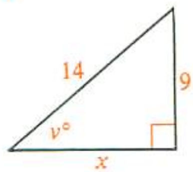


10. a. Use the Pythagorean Theorem to find the value of x in radical form.

b. Use the sine table to find $v°$.

c. Use the cosine table and the answer in (b) to find the value of x.

d. Are the x values from (a) and (c) in reasonable agreement?



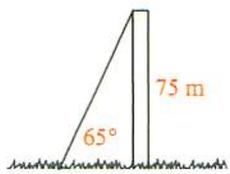
B **11.** The base of an isosceles triangle is 32 cm long and the legs are 24 cm long. Find the measure of a base angle. (*Hint:* Draw the altitude to the base.)

12. The base of an isosceles triangle is 42 cm long and the legs are 25 cm long. Find the measure of the vertex angle.

13. An isosceles triangle with legs 60 cm long has a 42° base angle. Find the lengths of the altitude and the base.

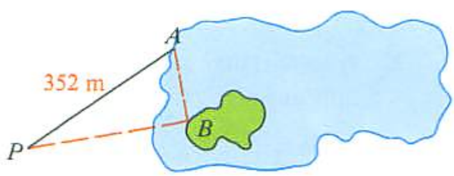
14. Points A, B, and C are three consecutive vertices of a regular decagon whose sides are 16 cm long. How long is diagonal $\overline{AC}$?

15. A guy wire is attached to the top of a 75 m tower and meets the ground at a 65° angle. How long is the wire?

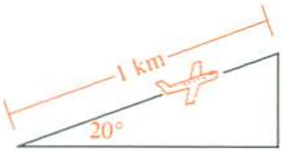


Ex. 15

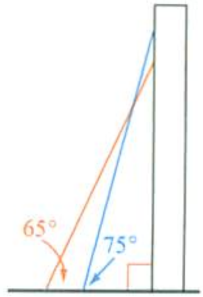
16. To find the distance from point A on the shore of a lake to point B on an island in the lake, surveyors locate point P with $m\angle PAB = 65$ and $m\angle APB = 25$. They find $PA = 352$ m. Find AB.



17. A certain jet is capable of a steady 20° climb. How much altitude does the jet gain when it moves 1 km through the air? Answer to the nearest 50 m.

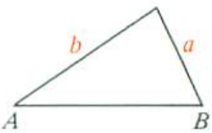


18. A 6 m ladder reaches higher up a wall when placed at a 75° angle than when placed at a 65° angle. How much higher, to the nearest tenth of a meter?



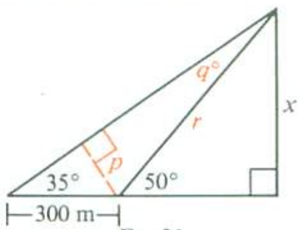
In Exercises 19 and 20, write the statements, but not the supporting reasons, of the proof.

C 19. Prove that in any triangle with acute angles A and B: $\frac{a}{\sin A} = \frac{b}{\sin B}$. (*Hint:* Draw a perpendicular from the third vertex to $\overline{AB}$. Label it p.)

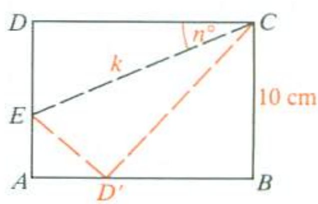


20. Prove: Where E is any acute angle, $(\sin E)^2 + (\cos E)^2 = 1$. (*Hint:* From any point on one side of $\angle E$, draw a perpendicular to the other side.)

21. The diagram in black is given. One way to determine the length x is to draw the red segment and compute the values, in the order named, of p, $q°$, r, and x. Find x.



Ex. 21

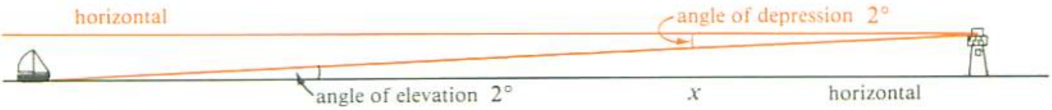


Ex. 22

22. A rectangular card is 10 cm wide. The card is folded so that the vertex D falls at point D' on $\overline{AB}$ as shown. Crease $\overline{CE}$ with length k makes an $n°$ angle with $\overline{CD}$. Show: $k = \frac{10}{\sin (2n)° \cos n°}$

6-7 Using Trigonometric Ratios

Suppose an operator at the top of a lighthouse sights a sailboat on a line that makes a 2° angle with a horizontal line. That angle is called an **angle of depression.** At the same time, a person in the boat must look 2° above the horizontal to see the top of the lighthouse. This is an **angle of elevation.** In this situation, the angle of elevation for one observer has the same measure as the angle of depression for the other observer.



If the top of the lighthouse is 25 m above sea level, the distance x between the boat and the base of the lighthouse can be found in these two ways:

Method 1

$$\tan 2^\circ = \frac{25}{x}$$
$$0.0349 \approx \frac{25}{x}$$
$$0.0349x \approx 25$$
$$x \approx 25 \div 0.0349$$
$$x \approx 716.3$$

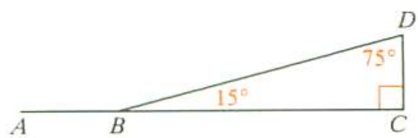
Method 2

$$\tan 88^\circ = \frac{x}{25}$$
$$28.6363 \approx \frac{x}{25}$$
$$25 \cdot 28.6363 \approx x$$
$$715.9 \approx x$$

Because the tangent values in the table are approximations, the two methods give slightly different answers. In practice, the angle measurement will not be exact, and the boat may be moving. In a case like this we can not claim high accuracy for our answer. A good answer would be: The boat is roughly 700 m from the lighthouse.

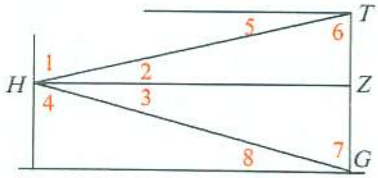
Classroom Exercises

1. $\overleftrightarrow{AC}$ is horizontal.
   a. For an observer at B, what is the angle of elevation of D?
   b. From D, what is the angle of depression of B?
   c. An observer at A measures the angle of elevation of D. Is the measure greater than 15 or less than 15?



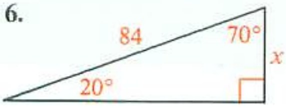
The lines shown are horizontal and vertical lines except for $\overleftrightarrow{HT}$ and $\overleftrightarrow{HG}$. Give the number of the angle and its special name when:

2. A person at H sights T.
3. A person at H sights G.
4. A person at T sights H.
5. A person at G sights H.



State two equations you could use to find the value of x.

6.



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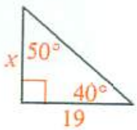

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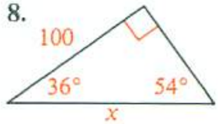
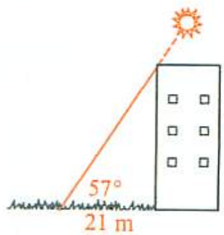


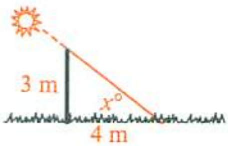
Written Exercises

Express lengths correct to the nearest integer and measures of angles correct to the nearest degree.

A 1. When the sun's angle of elevation is 57°, a building casts a shadow 21 m long. How high is the building?



2. At a certain time, a 3 m vertical pole casts a 4 m shadow. What is the angle of elevation of the sun?



In Exercises 3–6, first draw a diagram.

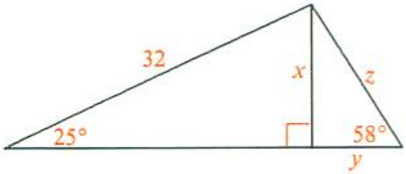
3. A kite is flying at an angle of elevation of about 40°. All 80 m of string have been let out. Ignoring the sag in the string, find the height of the kite to the nearest 10 m.

4. An advertising blimp hovers over a stadium at an altitude of 125 m. The pilot sights a tennis court at an 8° angle of depression. Find the ground distance in a straight line between the stadium and the tennis court. (*Note:* In an exercise like this one, an answer saying *about . . . hundred meters* is sensible.)

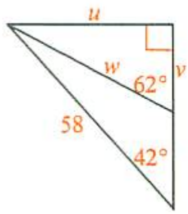
5. A rectangle is 20 m long and 10 m wide.
   a. Find the measure of the angle a diagonal makes with one of the longer sides.
   b. Use trigonometry and your answer from (a) to find the length of a diagonal.
   c. Use the Pythagorean Theorem to find the length of a diagonal.

6. Martha is 180 cm tall and her daughter Heidi is just 90 cm tall. Who casts the longer shadow, Martha when the sun is 70° above the horizon or Heidi when the sun is 35° above the horizon? How much longer?

B 7. Find x, then y and z.



8. Find u, then v and w.



9. Find the length of a diagonal of a regular pentagon with side 20.

10. Points A, B, C, and D are four consecutive vertices of a regular 9-gon (nonagon) with sides 25 mm long. Find the length of $\overline{AD}$.

11. The steepness of a hill is sometimes measured by the grade. A grade of 1 in 4 means that the hill rises one unit for every 4 horizontal units.

   a. For a grade of 1 in 4, what is the measure of $\angle A$, the angle the hill makes with the horizontal?

   b. The force of gravity pulling an object down the hill is its weight multiplied by the sine of $\angle A$. On a 1 in 4 grade, what is the force on a 2500 lb car?

   c. Could you push the car up the hill?

Given the value of one trigonometric ratio, find the exact value of the other two ratios.

Example $\sin A = \frac{7}{9}$

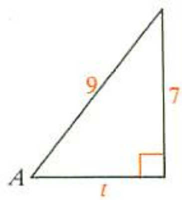
Solution (1) Sketch a right triangle with a leg and hypotenuse in the ratio 7:9.

(2) Use the Pythagorean Theorem to find the third side.

$t^2 + 49 = 81 \qquad t^2 = 32 \qquad t = 4\sqrt{2}$

(3) $\tan A = \frac{7}{t} = \frac{7}{4\sqrt{2}} = \frac{7}{4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{8}$

$\cos A = \frac{t}{9} = \frac{4\sqrt{2}}{9}$



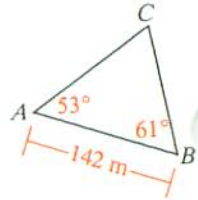
12. $\sin A = \frac{11}{61}$

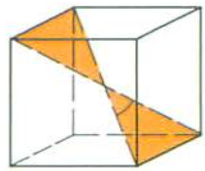
13. $\tan A = \frac{5}{8}$

C 14. $\cos A = \frac{j}{k}$

15. $\tan A = \frac{2uv}{u^2 - v^2}$

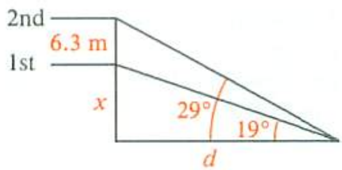
16. A surveyor wants to find the distance from points A and B to an inaccessible point C. Point C can be sighted from both A and B. The surveyor measures $\overline{AB}$, $\angle A$, and $\angle B$, with the results shown. Find AC and BC. (*Hint:* Use Exercise 19 on page 276.)

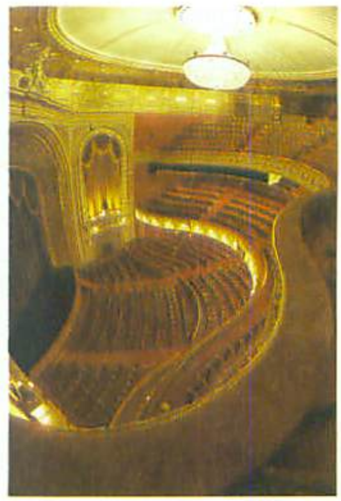




17. Find the acute angle at which two diagonals of a cube intersect.

★18. From the stage of a theater, the angle of elevation of the first balcony is 19°. The angle of elevation of the second balcony, 6.3 m directly above the first, is 29°. How high above stage level is the first balcony? (*Hint:* Use tan 19° and tan 29° to write two equations involving x and d. Solve for d, then find x.)





Application

PASSIVE SOLAR DESIGN

Passive solar homes are designed to let the sun heat the house during the winter, but to prevent the sun from heating the house during the summer. Because the Earth's axis is not perpendicular to the *ecliptic* (the plane of the Earth's orbit around the sun), the sun is lower in the sky in the winter than it is in the summer.

From the latitude of the homesite the architect can determine the elevation angle of the sun (the angle at which a person has to look up from the horizontal to see the sun) during the winter and during the summer. The architect can then design an overhang for windows that will let sunlight in the windows during the winter, but will shade the windows during the summer.

The Earth's axis makes an angle of $23\frac{1}{2}°$ with a perpendicular to the ecliptic plane. So for places in the northern hemisphere between the Tropic of Cancer and the Arctic Circle, the angle of elevation of the sun at noon on the longest day of the year, at the summer solstice, is $90° -$ the latitude $+ 23\frac{1}{2}°$. Its angle of elevation at noon on the shortest day, at the winter solstice, is $90° -$ the latitude $- 23\frac{1}{2}°$. For example, in Terre Haute, Indiana, at latitude $39\frac{1}{2}°$ north, the angle of elevation of the sun at noon on the longest day is 74° $(90 - 39\frac{1}{2} + 23\frac{1}{2} = 74)$, and at noon on the shortest day it is 27° $(90 - 39\frac{1}{2} - 23\frac{1}{2} = 27)$.

Exercises

Find the angle of elevation of the sun at noon on the longest day and at noon on the shortest day in the following cities. The approximate north latitudes are in parentheses.

1. Seattle, Washington ($47\frac{1}{2}°$)

2. Chicago, Illinois (42°)

3. Houston, Texas (30°)

4. Los Angeles, California (34°)

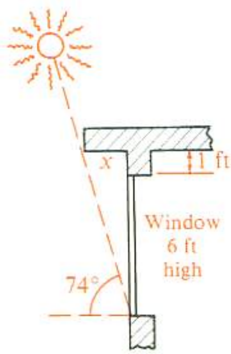
5. Nome, Alaska ($64\frac{1}{2}°$)

6. Miami, Florida (26°)

7. For a city south of the Tropic of Cancer, such as San Juan, Puerto Rico ($18\frac{1}{2}°$N), the formula gives a summer solstice angle greater than 90°. What does this mean?

8. For a place north of the Arctic Circle, such as Prudhoe Bay, Alaska (70°N), the formula gives a negative value for the angle of elevation of the sun at noon at the winter solstice. What does this mean?

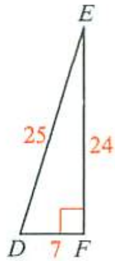
9. An architect is designing a passive solar house to be located in Terre Haute, Indiana. The diagram shows a cross-section of a wall that will face south. How long must the overhang x be to shade the entire window at noon at the summer solstice?



10. If the overhang has the length found in Exercise 9, how much of the window will be in the sun at noon at the winter solstice?

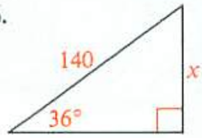
Self-Test 3

Exercises 1-5 refer to the diagram at the right.



1. $\tan D = \frac{?}{?}$
2. $\cos D = \frac{?}{?}$
3. $\sin D = \frac{?}{?}$
4. $\tan E = \frac{?}{?}$
5. To the nearest degree, $m\angle D =$ __?__. (Use the table on page 271.)

Find the value of x to the nearest integer. (Use the table on page 271.)

6.


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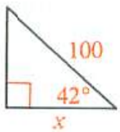

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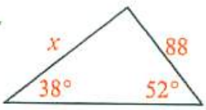
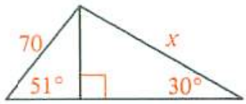
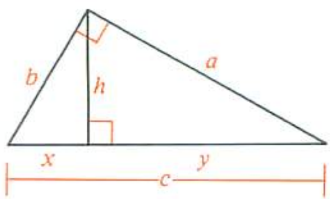


9.


10. A flagpole has a height of 14 m. From a point on the ground 100 m from the foot of the pole, what is the angle of elevation of the top?

Chapter Summary

1. When $\frac{a}{x} = \frac{x}{d}$, x is the geometric mean between a and d.
2. A right triangle is shown with the altitude drawn to the hypotenuse.

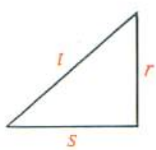


   a. The two triangles formed are similar to the original triangle and to each other.

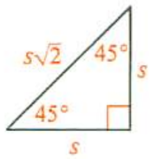
   $\frac{x}{h} = \frac{h}{y}$ $\qquad$ $\frac{c}{b} = \frac{b}{x}$ $\qquad$ $\frac{c}{a} = \frac{a}{y}$

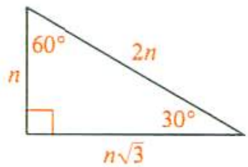
   b. Pythagorean Theorem: $c^2 = a^2 + b^2$

3. The longest side of the triangle shown is t.
   If $t^2 = r^2 + s^2$, the triangle is a right triangle.
   If $t^2 > r^2 + s^2$, the triangle is obtuse.
   If $t^2 < r^2 + s^2$, the triangle is acute.



4. The sides of a 45°-45°-90° triangle and the sides of a 30°-60°-90° triangle are related as shown.

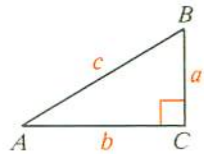




5. In the right triangle shown:

$\tan A = \frac{a}{b}$ $\quad\sin A = \frac{a}{c}$ $\quad\cos A = \frac{b}{c}$

The tangent, sine, and cosine ratios are useful in solving problems involving right triangles.



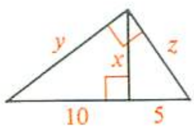
Chapter Review

1. Find the geometric mean between 12 and 3. **6-1**

2. $x =$ __?__

3. $y =$ __?__

4. $z =$ __?__



5. The legs of a right triangle are 3 and 6. Find the length of the hypotenuse. **6-2**

6. A rectangle has sides 10 and 8. Find the length of a diagonal.

7. The diagonal of a square has length 14. Find the length of a side.

8. The legs of an isosceles triangle are 10 units long and the altitude to the base is 8 units long. Find the length of the base.

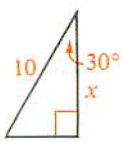
Tell whether a triangle formed with sides having the lengths named is acute, right, or obtuse. If a triangle can't be formed, write *not possible*.

9. 4, 5, 6 | 10. 8, 8, 17 **6-3**

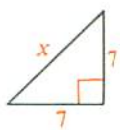
11. 11, 60, 61 | 12. $2\sqrt{3}$, $3\sqrt{2}$, 6

Find the value of x.

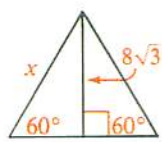
13.



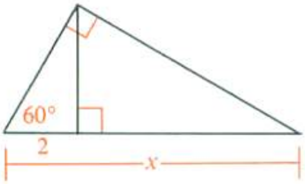
14.



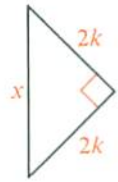
15. **6-4**



16. Find the value of x.



17. Express x in terms of k.



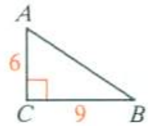
Complete. For Exercises 20, 21, 24, and 25 use the table on page 271.

18. $\tan A =$ __?__ 6-5

19. $\tan B =$ __?__

20. $\tan 72° \approx$ __?__

21. $\tan$ __?__ ≈ 0.4452

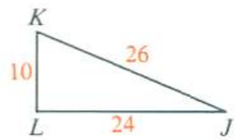


22. $\sin J =$ __?__ 6-6

23. $\cos K =$ __?__

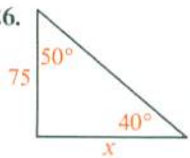
24. $\cos$ __?__ ≈ 0.2588

25. $\sin 43° \approx$ __?__

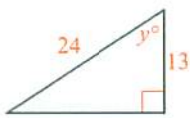


Find x correct to the nearest integer. Find y correct to the nearest degree. 6-7

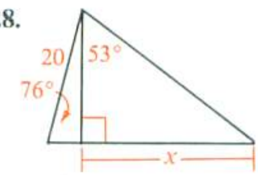
26.



27.



28.



Chapter Test

Find the geometric mean between the numbers.

1. 5 and 20 **2.** 6 and 8

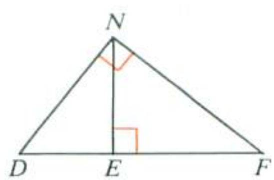
In the diagram, $\angle DNF$ is a right angle and $\overline{NE} \perp \overline{DF}$.

3. $\triangle DNF \sim \triangle$ __?__, and $\triangle DNF \sim \triangle$ __?__.

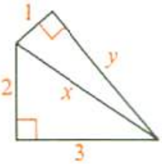
4. NE is the geometric mean between __?__ and __?__.

5. NF is the geometric mean between __?__ and __?__.

6. If $DE = 10$ and $EF = 15$, then $ND =$ __?__.



7. Find the values of x and y.

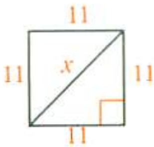


Tell whether a triangle formed with sides having the lengths named is acute, right, or obtuse. If a triangle can't be formed, write *not possible*.

8. 3, 4, 8 **9.** 11, 12, 13 **10.** 7, 7, 10 **11.** $\frac{3}{5}$, $\frac{4}{5}$, 1

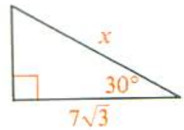
Find the value of x.

12.



13.

14.



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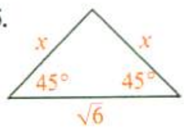

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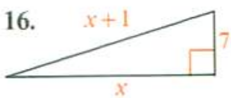

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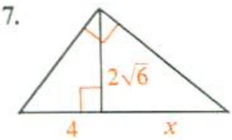
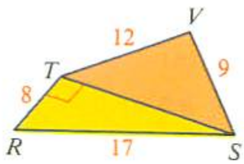
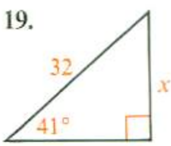


18. In the diagram, $\angle RTS$ is a right angle; $\overline{RT}$, $\overline{RS}$, $\overline{VT}$, and $\overline{VS}$ have the lengths shown.
 a. What kind of angle is $\angle V$?
 b. Explain your answer to part (a).

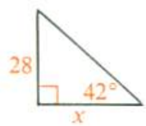


In Exercises 19–24 use the table on page 271. Find lengths correct to the nearest integer and angle measures correct to the nearest degree.

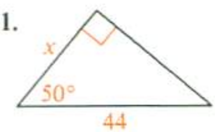
19.



20.



21.



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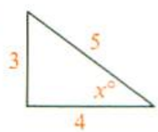

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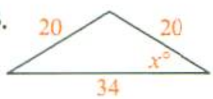

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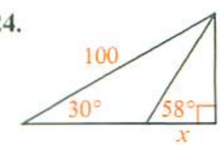


Preparing for College Entrance Exams

Strategy for Success

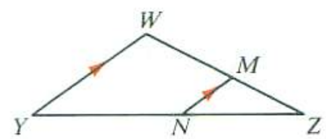
Problems in college entrance exams often involve right triangles. You can prepare for the exam by learning the common right-triangle lengths listed on page 258. Also, keep in mind that if a, b, and c are the lengths of the sides of a right triangle, then for any $x > 0$, ax, bx, and cx are also lengths of sides of a right triangle.

Indicate the best answer by writing the appropriate letter.

1. In $\triangle ABC$, $m\angle A : m\angle B : m\angle C = 2:5:5$. $m\angle B =$
(A) 75 **(B)** 60 **(C)** 30 **(D)** 40 **(E)** 100

2. The proportion $\frac{t}{z} = \frac{m}{k}$ is *not* equivalent to:
(A) $\frac{t-z}{z} = \frac{m-k}{k}$ **(B)** $\frac{k}{z} = \frac{m}{t}$ **(C)** $\frac{t}{m} = \frac{k}{z}$ **(D)** $tk = mz$ **(E)** $\frac{z}{t} = \frac{k}{m}$

3. If $\triangle ABC \sim \triangle DEF$, which statement is not necessarily true?
(A) $\angle C \cong \angle F$ **(B)** $\overline{BC} \cong \overline{EF}$ **(C)** $\frac{AB}{BC} = \frac{DE}{EF}$
(D) $m\angle A + m\angle E = m\angle B + m\angle D$ **(E)** $AC \cdot DE = DF \cdot AB$



4. If $ZY = 2x + 9$, $ZM = 10$, $ZN = x + 3$, and $MW = x$, then $x =$
(A) $2 + \sqrt{34}$ **(B)** -12 **(C)** 12 **(D)** 5 **(E)** -5

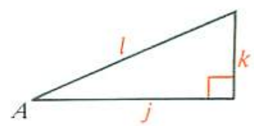
5. $\overrightarrow{BD}$ bisects $\angle ABC$ and D lies on $\overline{AC}$. If $AB = 6$, $BC = 14$, and $AC = 14$, find AD.
(A) 6 **(B)** 8.4 **(C)** 9.8 **(D)** 7 **(E)** 4.2

6. Find the geometric mean of $2x$ and $2y$.
(A) $2\sqrt{xy}$ **(B)** $\sqrt{2xy}$ **(C)** $2\sqrt{x+y}$ **(D)** $\sqrt{2(x+y)}$ **(E)** $4xy$

7. If $XY = 8$, $YZ = 40$, and $XZ = 41$, then:
(A) $\triangle XYZ$ is acute **(B)** $\triangle XYZ$ is right **(C)** $\triangle XYZ$ is obtuse
(D) $m\angle Y < m\angle Z$ **(E)** no $\triangle XYZ$ is possible

8. A rhombus contains a $120°$ angle. Find the ratio of the length of the longer diagonal to the length of the shorter diagonal.
(A) $\sqrt{3}:1$ **(B)** $\sqrt{3}:3$ **(C)** $\sqrt{2}:1$ **(D)** $\sqrt{2}:2$ **(E)** cannot be determined

9. $k =$



(A) $j \sin A$ **(B)** $j \tan A$ **(C)** $\frac{l}{\sin A}$
(D) $l \cos A$ **(E)** $l \tan A$

10. The legs of an isosceles triangle have length 4 and the base angles have measure 65. If $\sin 65° \approx 0.91$, $\cos 65° \approx 0.42$, and $\tan 65° \approx 2.14$, then the approximate length of the base of the triangle is:
(A) 1.7 **(B)** 1.9 **(C)** 3.4 **(D)** 3.6 **(E)** 4.4

Cumulative Review: Chapters 1-6

True-False Exercises

Write T or F to indicate your answer.

A 1. Through any three points there is exactly one plane.

2. If $AX = XB$, then X must be the midpoint of $\overline{AB}$.

3. Definitions may be used to justify statements in a proof.

4. If a line and a plane are parallel, then the line is parallel to every line in the plane.

5. Every isosceles trapezoid contains two pairs of congruent angles.

6. When two parallel lines are cut by a transversal, any two angles formed are either congruent or supplementary.

7. If the sides of one triangle are congruent to the corresponding sides of another triangle, then the corresponding angles must also be congruent.

8. The geometric mean of 18 and 50 is 34.

B 9. In quadrilateral $WXYZ$, if $WX = 25$, $XY = 25$, $YZ = 20$, $ZW = 16$, and $WY = 20$, then $\overline{WY}$ divides the quadrilateral into two similar triangles.

10. The bisector of the 60° angle of a 30°-60°-90° triangle separates the opposite side into segments with the ratio 1:2.

11. If a quadrilateral has two pairs of supplementary angles, then it must be a parallelogram.

12. In $\triangle PQR$, $m\angle P = m\angle R = 50$. If T lies on $\overline{PR}$ and $m\angle PQT = 42$, then $PT < TR$.

13. If a line parallel to one side of a triangle intersects the other two sides, then the triangle formed is similar to the given triangle.

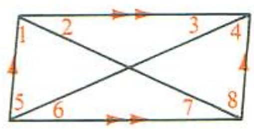
14. In any right triangle, the sine of one acute angle is equal to the cosine of the other acute angle.

15. If the diagonals of a quadrilateral bisect each other and are congruent, then the quadrilateral must be a square.

Multiple-Choice Exercises

Indicate the best answer by writing the appropriate letter.

A 1. Which pair of angles must be congruent?



a. $\angle 1$ and $\angle 4$ b. $\angle 2$ and $\angle 3$
c. $\angle 2$ and $\angle 4$ d. $\angle 4$ and $\angle 5$
e. $\angle 2$ and $\angle 8$

2. Which of the following can be the lengths of the sides of a triangle?
a. 3, 7, 10 b. 3, 7, 11 c. 0.5, 7, 7 d. $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}$ e. 1, 3, 5

3. Which of the following can be the lengths of the sides of a right triangle?
a. 2, 3, 4 b. 6, 8, $\sqrt{14}$ c. $\frac{1}{2}, \frac{1}{2}, 1$ d. $\sqrt{3}, \sqrt{5}, \sqrt{15}$ e. none of these

4. If $\triangle ABC \cong \triangle NDH$, then it is also true that:
a. $\angle B \cong \angle H$ b. $\angle A \cong \angle H$ c. $\overline{AB} \cong \overline{HD}$ d. $\overline{CA} \cong \overline{HN}$ e. $\triangle CBA \cong \triangle DHN$

5. If $PQRS$ is a parallelogram, which of the following *must* be true?
a. $PQ = QR$ b. $PQ = RS$ c. $PR = QS$ d. $\overline{PR} \perp \overline{QS}$ e. $\angle Q \cong \angle R$

B 6. If the diagonals of a rhombus have lengths 18 and 24, then the sides of the rhombus have length:
a. 15 b. 30 c. $6\sqrt{7}$ d. 12 e. cannot be determined

7. The altitude to the hypotenuse of a 30°-60°-90° triangle has length 6. The longer leg has length:
a. $2\sqrt{3}$ b. $6\sqrt{3}$ c. $4\sqrt{3}$ d. $8\sqrt{3}$ e. 12

8. If a, b, c, and d are coplanar lines such that $a \perp b$, $c \perp d$, and $b \parallel c$, then:
a. $a \perp d$ b. $b \parallel d$ c. $a \parallel d$ d. $a \parallel c$ e. none of these

Always-Sometimes-Never Exercises

Write A, S, or N to indicate your choice.

A 1. An angle _?_ has a complement.

2. Two vertical angles are _?_ adjacent.

3. Two parallel lines are _?_ coplanar.

4. A scalene triangle is _?_ equiangular.

5. A rectangle is _?_ a rhombus.

6. A regular polygon is _?_ equilateral.

7. If a conditional is false, then its converse is _?_ false.

8. The sine of an acute angle is _?_ greater than 1.

9. If $\overline{RS} \cong \overline{MN}$, $\overline{ST} \cong \overline{NO}$, and $\angle R \cong \angle M$, then $\triangle RST$ and $\triangle MNO$ are _?_ congruent.

10. The HL method is _?_ appropriate for proving that two acute triangles are congruent.

11. If $AX = BX$, $AY = BY$, and points A, B, X, Y are coplanar, then $\overline{AB}$ and $\overline{XY}$ are _?_ perpendicular.

B 12. The diagonals of a trapezoid are _?_ perpendicular.

13. If $\triangle JKL \cong \triangle NET$ and $\overline{NE} \perp \overline{ET}$, then it is _?_ true that $LJ < TE$.

14. Two perpendicular lines are _?_ both parallel to a third line.

15. If $AB + BC > AC$, then A, B, and C are _?_ collinear points.

16. Two equilateral octagons are _?_ similar.

17. Given the lengths of the legs of a right triangle, it is _?_ possible to find approximations for the measures of the angles of the triangle.

18. A triangle with sides of length $x - 1$, x, and x is _?_ an obtuse triangle.

Algebraic Exercises

In Exercises 1-12 find the value of x.

A 1. The lengths of the legs of an isosceles triangle are $7x - 13$ and $2x + 17$.

2. An angle and its complement have the measures $x + 38$ and $2x - 5$.

3. Consecutive angles of a parallelogram have the measures $6x$ and $2x + 20$.

4. The measures of the angles of a triangle are $x + 12$, $2x - 7$, and $3x + 1$.

5. On a number line, R and S have coordinates -8 and x, and the midpoint of $\overline{RS}$ has coordinate -1.

6. Two vertical angles have measures $x^2 + 18x$ and $x^2 + 54$.

7. The measures of the angles of a quadrilateral are x, $x + 4$, $x + 8$, and $x + 12$.

8. The hypotenuse of a $30°$-$60°$-$90°$ triangle has length $8\sqrt{3}$ and the longer leg has length $4x$.

9. A trapezoid has bases of length x and $x + 8$ and a median of length 15.

10. $\frac{3x - 1}{4x + 2} = \frac{2}{3}$ 11. $\frac{5}{8} = \frac{x - 1}{6}$ 12. $\frac{x}{x + 4} = \frac{x + 3}{x + 9}$

B 13. The measure of a supplement of an angle is 8 more than three times the measure of a complement. Find the measure of the angle.

14. A triangle with perimeter 64 cm has sides with lengths in the ratio 4:5:7. Find the length of each side.

15. In $\triangle XYZ$, $XY = YZ$. If $\frac{m\angle X}{m\angle Y} = \frac{5}{2}$, find the numerical measure of $\angle Z$.

16. In a regular polygon, the ratio of the measure of an exterior angle to the measure of an interior angle is 2:13. How many vertices does the polygon have?

17. In $\triangle RST$, $RS = 8$, $ST = 9$, and $RT = 12$. $\overrightarrow{RU}$ bisects $\angle SRT$ and U lies on $\overline{ST}$. $SU = \underline{\ ?\ }$.

18. The sides of an obtuse triangle have lengths x, $2x + 2$, and $2x + 3$. $\underline{\ ?\ } < x < \underline{\ ?\ }$.

19. The sides of a parallelogram have lengths 12 cm and 15 cm. Find the lengths of the sides of a similar parallelogram with perimeter 90 cm.

20. The length of a diagonal of a rectangle is 10, and the perimeter of each right triangle formed is 24. Find the length and width of the rectangle.

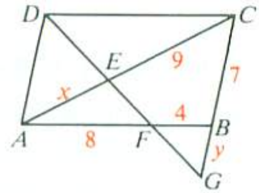
In Exercises 21-23 right $\triangle XYZ$ has hypotenuse $\overline{XZ}$.

21. If $\cos X = \frac{7}{10}$ and $XZ = 24$, then to the nearest integer $XY = \underline{\ ?\ }$.

22. If $XY = 10$ and $YZ = 15$, then to the nearest degree $m\angle X = \underline{\ ?\ }$ and $m\angle Z = \underline{\ ?\ }$. (Use the table on page 271.)

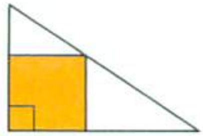
23. If $\overline{YM}$ is an altitude of $\triangle XYZ$, $YZ = 18$, and $XZ = 24$, then $XM = \underline{\ ?\ }$.

24. In the diagram, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{GC}$. Find the values of x and y.



C 25. The sides of a triangle have lengths $x + y$, $x - y$, and $2\sqrt{xy}$. Is the triangle acute, right, or obtuse?

26. A piece of plywood is in the shape of a right triangle with legs of length 12 cm and 8 cm. A square is cut from the triangle as shown. Find the length of each side of the square.



Completion Exercises

Complete each statement in the best way.

A 1. If $\overrightarrow{YW}$ bisects $\angle XYZ$ and $m\angle WYX = 60$, then $m\angle XYZ =$ __?__.

2. The acute angles of a right triangle are __?__.

3. A supplement of an acute angle is a(n) __?__ angle.

4. Adjacent angles formed by __?__ lines are congruent.

5. The measure of each interior angle of a regular pentagon is __?__.

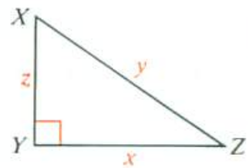
6. In an isosceles right triangle, the ratio of the length of a leg to the length of the hypotenuse is __?__.

7. In $\triangle ABC$ and $\triangle DEF$, $\angle A \cong \angle D$ and $\angle B \cong \angle E$. $\triangle ABC$ and $\triangle DEF$ must be __?__.

8. If $x = 8$, $y = 10$, and $z = 6$, then $\sin X =$ __?__, $\cos Z =$ __?__, and $\tan Z =$ __?__.

9. If $m\angle X = 50$ and $z = 20$, then to the nearest integer $x \approx$ __?__. (Use the table on page 271.)

10. The ratio of the measures of the acute angles of a right triangle is 3:2. The measure of the smaller acute angle is __?__.



Exs. 8, 9

B 11. If $\frac{r}{s} = \frac{t}{u}$, then $\frac{r+s}{t+u} = \frac{?}{?}$.

12. An isosceles triangle has legs of 10 cm and a vertex angle of measure 120. The length of the base is __?__.

13. If $\sin B = \frac{8}{17}$, then $\cos B =$ __?__.

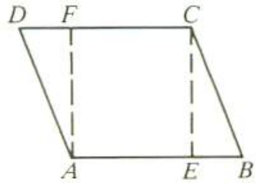
14. When the midpoints of the sides of a rhombus are joined in order, the resulting quadrilateral is best described as a __?__.

15. If a tree is 20 m high and the distance from point P on the ground to the base of the tree is also 20 m, then the angle of elevation of the top of the tree from point P is __?__.

Proof Exercises

A

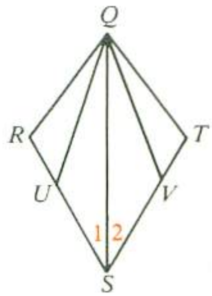
1. Given: $\overline{AD} \cong \overline{BC}$; $\overline{AD} \parallel \overline{BC}$
   Prove: $\angle D \cong \angle B$
2. Given: $ABCD$ is a ▱; $\overline{CE} \perp \overline{AB}$; $\overline{AF} \perp \overline{CD}$
   Prove: $\overline{BE} \cong \overline{DF}$
3. Given: $\triangle DAF \cong \triangle BCE$; $CD = AB$
   Prove: $ABCD$ is a ▱.
4. Given: $\overline{DC} \parallel \overline{AB}$; $\overline{CE} \perp \overline{AB}$; $\overline{AF} \perp \overline{AB}$
   Prove: $AECF$ is a rectangle.



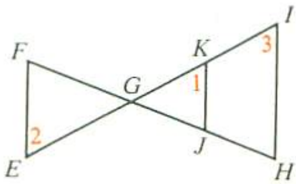
5. Given: $\overline{SU} \cong \overline{SV}$; $\angle 1 \cong \angle 2$
   Prove: $\overline{UQ} \cong \overline{VQ}$
6. Given: $\overrightarrow{QS}$ bisects $\angle RQT$; $\angle R \cong \angle T$
   Prove: $\overrightarrow{SQ}$ bisects $\angle RST$.

B

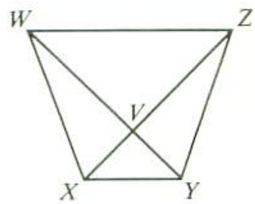
7. Given: $\triangle QRU \cong \triangle QTV$; $US = VS$
   Prove: $\triangle QRS \cong \triangle QTS$
8. Given: $\overline{QS}$ bisects $\angle UQV$ and $\angle USV$; $\angle R \cong \angle T$
   Prove: $\overline{RQ} \cong \overline{TQ}$



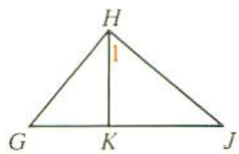
9. Given: $\overline{EF} \parallel \overline{JK}$; $\overline{JK} \parallel \overline{HI}$
   Prove: $\triangle EFG \sim \triangle IHG$
10. Given: $\frac{JG}{HG} = \frac{KG}{IG}$; $\angle 1 \cong \angle 2$
    Prove: $\overline{EF} \parallel \overline{HI}$



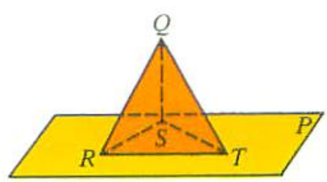
11. Given: $XZ = YW$; $VZ = VW$
    Prove: $XW = YZ$
12. Given: $\overline{XW} \cong \overline{YZ}$; $\angle XWZ \cong \angle YZW$
    Prove: $\triangle XVW \cong \triangle YVZ$
13. Given: $WXYZ$ is an isosceles trapezoid with $\overline{XW} \cong \overline{YZ}$.
    Prove: $\overline{XZ} \cong \overline{YW}$
14. Given: $VW = VZ$; $VX = VY$
    Prove: $XY \cdot VW = WZ \cdot VX$



15. Given: $\angle G \cong \angle 1$
    Prove: JH is the geometric mean between JK and JG.
16. Given: $\overline{GH} \perp \overline{JH}$; $\overline{HK} \perp \overline{GJ}$
    Prove: $(GH)^2 - (GK)^2 = GK \cdot KJ$



17. Given: Points R, S, and T lie in plane P;
    $\overline{QS} \perp$ plane P; $\triangle QRT$ is equilateral.
    Prove: $\angle SRT \cong \angle STR$



The photograph below, which shows meshed gears, suggests a number of ideas discussed in this chapter—for example, tangent circles, concentric circles, arcs of a circle, and secants of a circle.

Circles

7