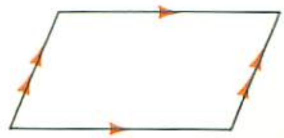
Parallelograms and Trapezoids

Objectives

1. Apply the definitions of a parallelogram and a trapezoid.
2. State and apply the theorems about properties of a parallelogram.
3. Prove that certain quadrilaterals are parallelograms.
4. Identify the special properties of a rectangle, a rhombus, and a square.
5. State and apply the theorems about the median of a trapezoid and the segment that joins the midpoints of two sides of a triangle.

4-1 Properties of Parallelograms

A **parallelogram** (▱) is a quadrilateral with both pairs of opposite sides parallel. The following theorems state some properties common to all parallelograms. Your proofs of these theorems (Exercises 17–19) will be based on what you have learned about parallel lines and congruent triangles.



Theorem 4-1

Opposite sides of a parallelogram are congruent.

Given: ▱$EFGH$

Prove: $\overline{EF} \cong \overline{HG}$; $\overline{FG} \cong \overline{EH}$

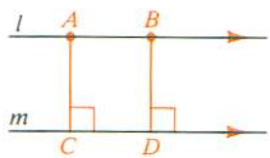
■ ***Plan for Proof:*** Draw $\overline{EG}$ to form triangles that can be proved congruent by ASA. Note that the pairs of alternate interior angles formed are $\angle 1$ and $\angle 2$, $\angle 3$ and $\angle 4$. After showing that the triangles are congruent, you can use corresponding parts to finish the proof.

Corollary

If two lines are parallel, then all points on one line are equidistant from the other line.

Given: $l \parallel m$; A and B are any points on l;
$\overline{AC} \perp m$; $\overline{BD} \perp m$

Prove: $AC = BD$



Proof:

Since $\overline{AB}$ and $\overline{CD}$ are contained in parallel lines, $\overline{AB} \parallel \overline{CD}$. Since $\overline{AC}$ and $\overline{BD}$ are both perpendicular to m, they are parallel. Thus $ABDC$ is a parallelogram, and opposite sides $\overline{AC}$ and $\overline{BD}$ are congruent.

Theorem 4-2

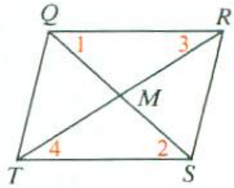
Opposite angles of a parallelogram are congruent.

Theorem 4-3

The diagonals of a parallelogram bisect each other.

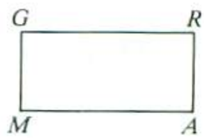
Given: $\square QRST$ with diagonals $\overline{QS}$ and $\overline{TR}$

Prove: $\overline{QS}$ and $\overline{TR}$ bisect each other.



■ ***Plan for Proof:*** You can prove that $\overline{QM} \cong \overline{MS}$ and $\overline{MR} \cong \overline{TM}$ by showing that they are corresponding parts of congruent triangles. Since you have $\overline{QR} \cong \overline{TS}$ by Theorem 4-1, you can use ASA to show that $\triangle QMR \cong \triangle SMT$.

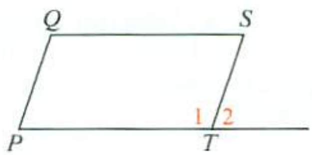
Classroom Exercises



1. Quad. *GRAM* is a parallelogram.
   a. Why is $\angle G$ supplementary to $\angle M$?
   b. Why is $\angle M$ supplementary to $\angle A$?
   c. Complete: Consecutive angles of a parallelogram are __?__, while opposite angles are __?__.

2. Suppose $\angle M$, in $\square GRAM$, is a right angle. What can you deduce about angles *G*, *R*, and *A*?

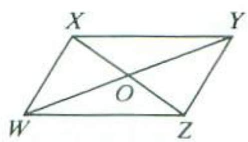
Find the measures of $\angle Q$, $\angle S$, $\angle 1$, and $\angle 2$. Quad. *PQST* is a parallelogram.



3. $m\angle P = 70$

4. $m\angle P = c$

In Exercises 5-7, quad. *WXYZ* is a parallelogram.



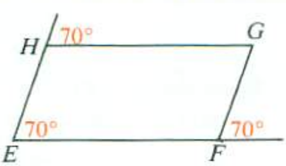
5. Name all pairs of parallel lines in the figure.

6. Name all pairs of congruent segments.

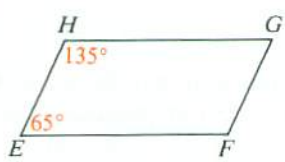
7. Name all pairs of congruent angles.

Must quad. *EFGH* be a parallelogram? Can it be a parallelogram? Explain.

8.

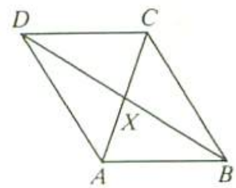


9.

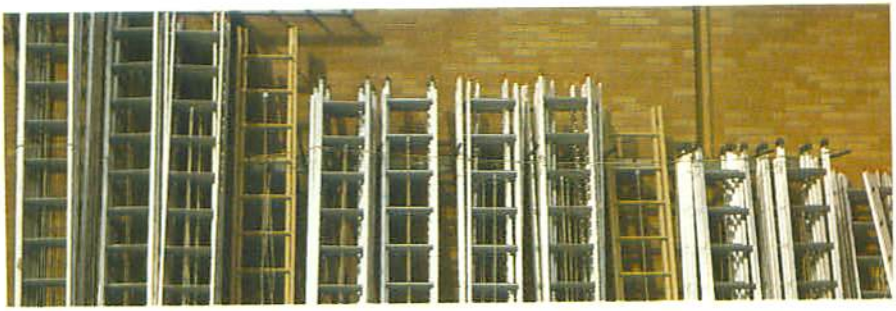


10.

Quad. $ABCD$ is a parallelogram. Name or state the principal theorem or definition that justifies the statement.

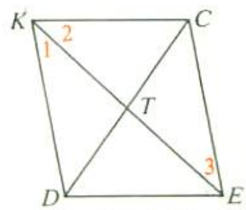


11. $\overline{AD} \parallel \overline{BC}$
12. $\angle ADX \cong \angle CBX$
13. $m\angle ABC = m\angle CDA$
14. $\overline{AD} \cong \overline{BC}$
15. $AX = \frac{1}{2}AC$
16. $DX = BX$
17. Draw a quadrilateral that isn't a parallelogram but does have two 60° angles.
18. What result of this section does each ladder suggest?



Written Exercises

Exercises 1-16 refer to $\square DECK$. Complete each statement in Exercises 1-8.


Exs. 1-16

A

1. If $DE = 10$, $KC =$ _?_.
2. If $DC = 18$, $DT =$ _?_.
3. If $m\angle EDK = 100$, $m\angle ECK =$ _?_.
4. If $m\angle DEC = 75$, $m\angle KDE =$ _?_.
5. If $m\angle 1 = 30$ and $m\angle 2 = 40$, $m\angle KCE =$ _?_.
6. If $m\angle 1 = 30$ and $m\angle 2 = 40$, $m\angle 3 =$ _?_.
7. If $m\angle 3 = 36$ and $m\angle 2 = 44$, $m\angle KDE =$ _?_.
8. If $DT = 7$ and $KT = 9$, $CD =$ _?_.

Find the value of x or y.

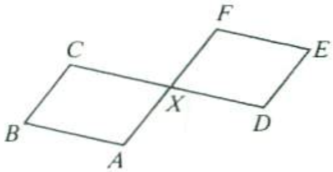
9. $DE = 5x$ and $KC = 3x + 12$
10. $DK = 2x + 5$ and $EC = 47 - 4x$
11. $ET = x + 3$ and $EK = 22$
12. $DT = \frac{1}{2}x$ and $TC = 10$
13. $m\angle KCE = 6y - 20$ and $m\angle EDK = 2y + 80$
14. $m\angle DEC = 80 - y$ and $m\angle DKC = y + 40$
15. $m\angle 1 = y + 10$, $m\angle 2 = 3y$, and $m\angle 3 = \frac{1}{2}y + 15$
16. $m\angle DEC = \frac{y}{4}$ and $m\angle ECK = \frac{y + 60}{2}$

17. Prove Theorem 4–1.

18. Prove Theorem 4–2. (Draw and label a figure. List what is given and what is to be proved.)

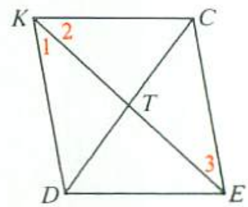
19. Prove Theorem 4–3.

20. Given: Quad. $ABCX$ is a ▱;
quad. $DXFE$ is a ▱.
Prove: $\angle B \cong \angle E$



Quad. *DECK* is a parallelogram. Complete.

B 21. If $KT = 2x + y$, $DT = x + 2y$, $TE = 12$, and $TC = 9$, then $x =$ _?_ and $y =$ _?_.



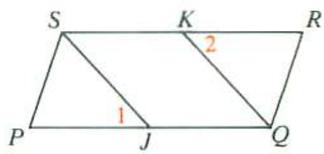
22. If $DE = x + y$, $EC = 12$, $CK = 2x - y$, and $KD = 3x - 2y$, then $x =$ _?_, $y =$ _?_, and the perimeter of ▱$DECK =$ _?_.

23. If $m\angle 1 = 4x$, $m\angle 2 = 3x$, and $m\angle 3 = x^2 - 60$, then $x =$ _?_ and $m\angle CED =$ _?_ (numerical answers).

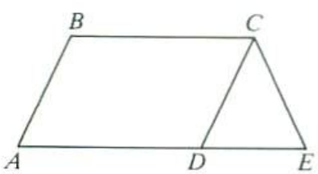
24. If $m\angle 1 = 20$, $m\angle 2 = x^2$, and $m\angle CED = 9x$, then $m\angle 2 =$ _?_ or $m\angle 2 =$ _?_ (numerical answers).

25. Given: ▱$PQRS$; $\overline{PJ} \cong \overline{RK}$
Prove: $\overline{SJ} \cong \overline{QK}$

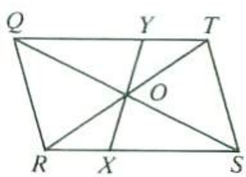
26. Given: ▱$JQKS$; $\overline{PJ} \cong \overline{RK}$
Prove: $\angle P \cong \angle R$



27. Given: $ABCD$ is a ▱; $\overline{CD} \cong \overline{CE}$
Prove: $\angle A \cong \angle E$

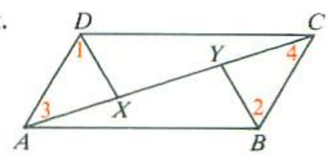


28. Given: $RSTQ$ is a ▱.
Prove: $\overline{OX} \cong \overline{OY}$



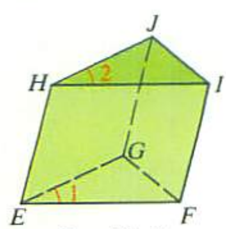
Find something interesting to prove. Then prove it.

29. Given: ▱$ABCD$; $\angle 1 \cong \angle 2$
Prove: _?_



30. Given: ▱$EFIH$; ▱$EGJH$; $\angle 1 \cong \angle 2$
Prove: _?_

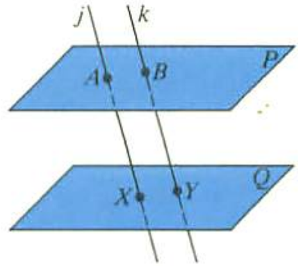
31. Given: $GF \neq JI$ and $GE \neq JH$
a. Can quadrilaterals $GFIJ$ and $EGJH$ be parallelograms? Explain.
b. Draw a diagram similar to that shown, but such that $EFIH$ is a parallelogram and it is clear that $GF \neq JI$ and $GE \neq JH$.



Exs. 30, 31

C **32. a.** Given: Plane $P \parallel$ plane Q; $j \parallel k$
Prove: $AX = BY$

b. State, in words, a theorem proved in part (a).



33. Prove: If a segment whose endpoints lie on opposite sides of a parallelogram passes through the midpoint of a diagonal, that segment is bisected by the diagonal.

★ **34.** Write a paragraph proof: The sum of the lengths of the segments drawn from any point in the base of an isosceles triangle perpendicular to the legs is equal to the length of the altitude drawn to one leg from the vertex opposite that leg.

4-2 Ways to Prove that Quadrilaterals Are Parallelograms

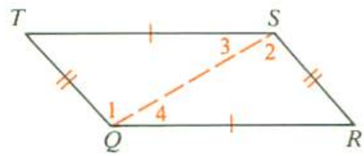
If both pairs of opposite sides of a quadrilateral are parallel, then by definition the quadrilateral is a parallelogram. The following theorems, whose proofs are left as exercises, will give you additional ways to prove that a quadrilateral is a parallelogram.

Theorem 4-4

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Given: $\overline{TS} \cong \overline{QR}$; $\overline{TQ} \cong \overline{SR}$

Prove: Quad. $QRST$ is a $\square$.



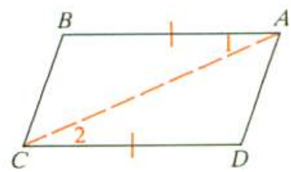
■ ***Plan for Proof:*** Draw $\overline{SQ}$ and prove $\triangle TSQ \cong \triangle RQS$. Then $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$, and opposite sides are parallel.

Theorem 4-5

If one pair of opposite sides of a quadrilateral are both congruent and parallel, then the quadrilateral is a parallelogram.

Given: $\overline{AB} \cong \overline{CD}$; $\overline{AB} \parallel \overline{CD}$

Prove: Quad. $ABCD$ is a $\square$.



■ ***Plan for Proof:*** Draw $\overline{AC}$ and prove $\triangle ABC \cong \triangle CDA$. Then $\overline{BC} \cong \overline{DA}$, and you can apply Theorem 4-4.

Theorem 4-6

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Theorem 4-7

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Five Ways to Prove that a Quadrilateral Is a Parallelogram

1. Show that *both* pairs of opposite sides are parallel.
2. Show that *both* pairs of opposite sides are congruent.
3. Show that *one* pair of opposite sides are both congruent and parallel.
4. Show that both pairs of opposite angles are congruent.
5. Show that the diagonals bisect each other.

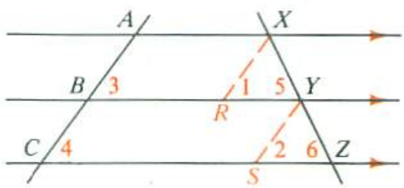
The proof of the following theorem is based on what you have learned about parallelograms.

Theorem 4-8

If three parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal.

Given: $\overleftrightarrow{AX} \parallel \overleftrightarrow{BY} \parallel \overleftrightarrow{CZ}$;
$\overline{AB} \cong \overline{BC}$

Prove: $\overline{XY} \cong \overline{YZ}$



Proof:

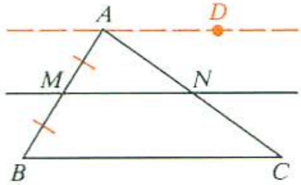
Through X and Y draw lines parallel to $\overleftrightarrow{AC}$. Then $AXRB$ and $BYSC$ are parallelograms, by the definition of a parallelogram. Since the opposite sides of a parallelogram are congruent, $\overline{XR} \cong \overline{AB}$ and $\overline{BC} \cong \overline{YS}$. It is given that $\overline{AB} \cong \overline{BC}$, so using the Transitive Property twice gives $\overline{XR} \cong \overline{YS}$. Parallel lines are cut by transversals to form the following pairs of congruent corresponding angles:

$\angle 1 \cong \angle 3 \qquad \angle 3 \cong \angle 4 \qquad \angle 4 \cong \angle 2 \qquad \angle 5 \cong \angle 6$

Then $\angle 1 \cong \angle 2$ (Transitive Property), and $\triangle XYR \cong \triangle YZS$ by AAS. Since $\overline{XY}$ and $\overline{YZ}$ are corresponding parts of these triangles, $\overline{XY} \cong \overline{YZ}$.

Corollary

A line that contains the midpoint of one side of a triangle and is parallel to another side bisects the third side.



Given: M is the midpoint of $\overline{AB}$;
$\overleftrightarrow{MN} \parallel \overline{BC}$

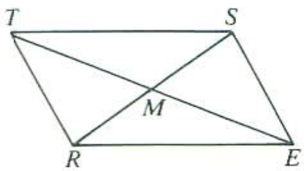
Prove: $\overleftrightarrow{MN}$ bisects $\overline{AC}$.

Proof:

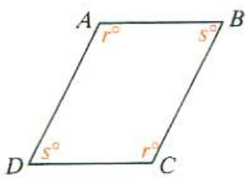
Let $\overleftrightarrow{AD}$ be the line through A parallel to $\overleftrightarrow{MN}$. Then $\overleftrightarrow{AD}$, $\overleftrightarrow{MN}$, and $\overleftrightarrow{BC}$ are three parallel lines that cut off congruent segments on transversal $\overleftrightarrow{AB}$. By Theorem 4-8 they also cut off congruent segments on $\overleftrightarrow{AC}$. Thus $\overline{AN} \cong \overline{NC}$ and $\overleftrightarrow{MN}$ bisects $\overline{AC}$.

Classroom Exercises

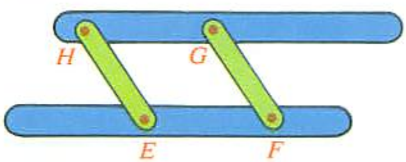
In each exercise decide whether the given information permits you to deduce that quad. *REST* is a parallelogram. If your answer is *yes*, state the definition or theorem that applies.



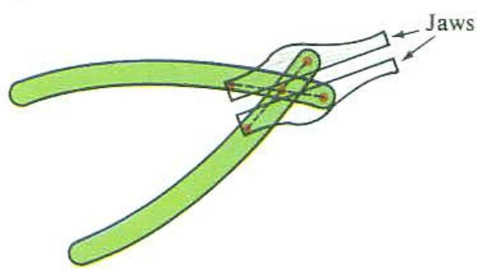
1. $\overline{TM} \cong \overline{EM}$
2. $TM = EM$; $RM = SM$
3. $\overline{TS} \parallel \overline{RE}$; $\overline{TS} \cong \overline{RE}$
4. $TS = RE$; $TR = SE$
5. $\overline{TS} \parallel \overline{RE}$; $\overline{TR} \parallel \overline{SE}$
6. $\overline{TS} \cong \overline{RE}$; $\overline{TS} \cong \overline{TR}$
7. Given: Quad. $ABCD$; $m\angle A = m\angle C = r$; $m\angle B = m\angle D = s$
   a. Tell why $2r + 2s = 360$.
   b. $r + s = \underline{\ ?\ }$
   c. Tell why $\overline{AB} \parallel \overline{CD}$ and why $\overline{AD} \parallel \overline{BC}$.
   d. Tell why quad. $ABCD$ must be a parallelogram.
   e. What theorem have you just proved?



8. *Parallel rulers,* used to draw parallel lines, are constructed so that $EF = HG$ and $HE = GF$. Since there are hinges at points E, F, G, and H, you can vary the distance between $\overline{HG}$ and $\overline{EF}$. Explain why $\overline{HG} \parallel \overline{EF}$.



9. The pliers shown are made in such a way that the jaws are always parallel. Explain.



10. Imagine quad. $WXYZ$, with sides $\overline{WX}$ and $\overline{ZY}$ congruent and sides $\overline{WZ}$ and $\overline{XY}$ parallel. Must $WXYZ$ be a parallelogram? Explain.

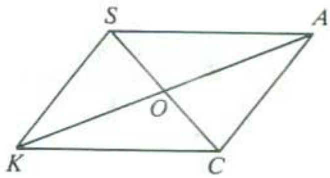
11. Imagine a quadrilateral with two pairs of sides congruent. Must the quadrilateral be a parallelogram? Explain.

Written Exercises

State the principal definition or theorem that enables you to deduce, from the information given, that quadrilateral $SACK$ is a parallelogram.

A

1. $\overline{SA} \parallel \overline{KC}$; $\overline{SK} \parallel \overline{AC}$
2. $\overline{SA} \cong \overline{KC}$; $\overline{SK} \cong \overline{AC}$
3. $\overline{SA} \cong \overline{KC}$; $\overline{SA} \parallel \overline{KC}$
4. $SO = \frac{1}{2}SC$; $KO = \frac{1}{2}KA$
5. $\angle SKC \cong \angle CAS$; $\angle KCA \cong \angle ASK$



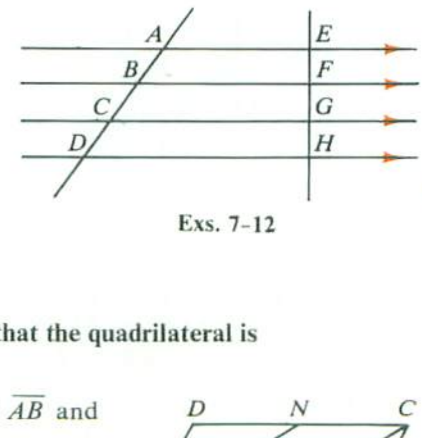
6. Suppose you know that $\triangle SOK \cong \triangle COA$. Explain how you could prove that quad. $SACK$ is a parallelogram.

$\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{CG}$ and $\overleftrightarrow{DH}$ are parallel, with $EF = FG = GH$. Complete.

7. If $AB = 5$, $AD =$ _?_.
8. If $AC = 12$, $CD =$ _?_.
9. If $AB = 5x$ and $BC = 2x + 12$, $x =$ _?_.
10. If $AC = 22 - x$ and $BD = 3x - 22$, $x =$ _?_.

B

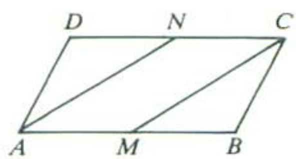
11. If $AB = 15$, $BC = 2x - y$, and $CD = x + y$, $x =$ _?_ and $y =$ _?_.
12. If $AB = 12$, $BC = 2x + 3y$, and $BD = 8x$, $x =$ _?_ and $y =$ _?_.



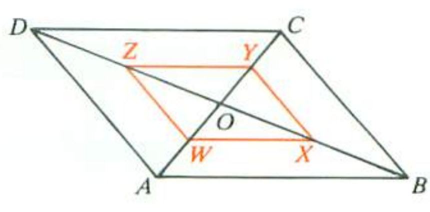
Exs. 7–12

In Exercises 13–15 explain briefly how you would prove that the quadrilateral is a parallelogram.

13. Given: ▱$ABCD$; M and N are the midpoints of $\overline{AB}$ and $\overline{DC}$.
    Prove: $AMCN$ is a ▱.
14. Given: ▱$ABCD$; $\overline{AN}$ and $\overline{CM}$ bisect $\angle A$ and $\angle C$.
    Prove: $AMCN$ is a ▱.

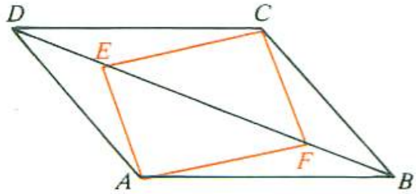


15. Given: ▱$ABCD$; W, X, Y, Z are midpoints of $\overline{AO}$, $\overline{BO}$, $\overline{CO}$, and $\overline{DO}$.
    Prove: $WXYZ$ is a ▱.

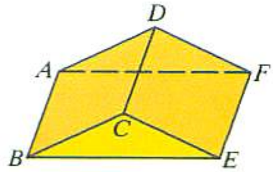


Explain briefly how you would prove that the quadrilateral is a parallelogram.

16. Given: $\square ABCD$; $DE = BF$
Prove: $AFCE$ is a $\square$.



17. Given: $\square ABCD$ and $\square CDFE$
Prove: $ABEF$ is a $\square$.



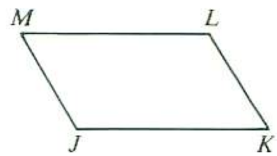
18. **a.** State Theorem 4-1 in if-then form.
b. Which theorem in this section is the converse of Theorem 4-1?

19. Prove Theorem 4-4.

20. Prove Theorem 4-5.

21. **a.** Prove Theorem 4-7.
b. Describe another way to prove Theorem 4-7.

What values must x and y have to make quad. $JKLM$ a parallelogram?

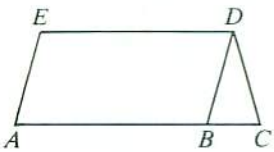


22. $ML = 42$, $LK = 26$, $KJ = 4x + y$, $JM = 3x - 2y$

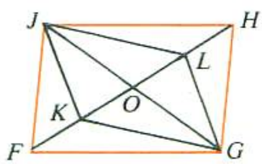
23. $ML = 5x - 3y$, $LK = x + y$, $KJ = 3x + y$, $JM = 33$

24. $ML = 2y - x$, $LK = 5$, $KJ = 2x - y$, $JM = x - \frac{y}{2}$

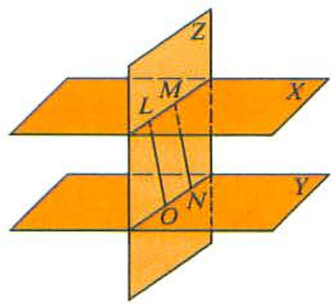
25. Given: $\overline{AE} \cong \overline{CD}$;
$\angle DBC \cong \angle C$;
$\angle A \cong \angle DBC$
Prove: Quad. $ABDE$ is a $\square$.



26. Given: $\square KGLJ$;
$FK = LH$
Prove: Quad. $FGHJ$ is a $\square$.



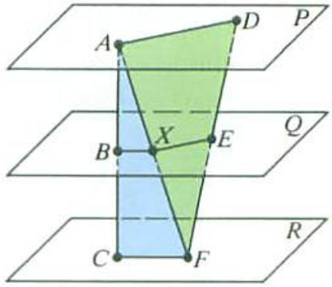
27. Given: Plane $X \parallel$ plane Y;
$\overline{LM} \cong \overline{ON}$
Prove: Quad. $LMNO$ is a $\square$.



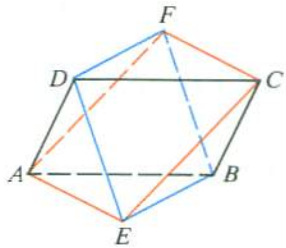
C **28.** Given: Parallel planes P, Q, and R cutting transversals $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$; $AB = BC$

Prove: $DE = EF$

(*Hint:* You can't assume that $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$ are coplanar. Draw $\overline{AF}$, cutting plane Q at X. Using the plane of $\overline{AC}$ and $\overline{AF}$, apply Theorems 2-1 and 4-8. Then use the plane of $\overline{AF}$ and $\overline{FD}$.)



29. Write a paragraph proof.
Given: ▱$ABCD$; ▱$BEDF$
Prove: $AECF$ is a ▱.
(*Hint:* A short proof is possible.)



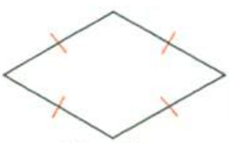
4-3 Special Parallelograms

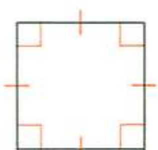
A quadrilateral with four right angles is a **rectangle.** Since both pairs of opposite angles are congruent, every rectangle is a parallelogram.

A quadrilateral with four congruent sides is a **rhombus.** Since both pairs of opposite sides are congruent, every rhombus is a parallelogram.

A quadrilateral with four right angles and four congruent sides is a **square.** Notice that every square is a special kind of rectangle, as well as a special kind of rhombus.


Rectangle


Rhombus


Square

As the photograph suggests, these shapes can be found in many everyday objects.

Since rectangles, rhombuses, and squares are parallelograms, they have all the properties of parallelograms. They also have the following special properties. Proofs of the theorems are left as exercises.

Theorem 4-9

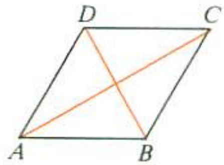
The diagonals of a rectangle are congruent.

Theorem 4-10

The diagonals of a rhombus are perpendicular.

Theorem 4-11

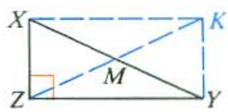
Each diagonal of a rhombus bisects two angles of the rhombus.

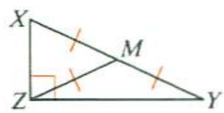


Example Given: $ABCD$ is a rhombus.
What can you conclude?

Solution $ABCD$ is a parallelogram, with all the properties of a parallelogram. Also:
By Theorem 4-10, $\overline{AC} \perp \overline{BD}$.
By Theorem 4-11, $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$;
$\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$.

The properties of rectangles lead to an interesting conclusion about any right triangle.





Begin with rt. $\triangle XZY$.

1. Draw lines to form rectangle $XZYK$. (How?)
2. Draw $\overline{ZK}$. $ZK = XY$ (Why?)
3. $\overline{ZK}$ and $\overline{XY}$ bisect each other. (Why?)
4. $MX = MY = MZ = MK$, by (2) and (3).

Since $MX = MY = MZ$, we have shown the following.

Theorem 4-12

The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Proofs of the next two theorems will be discussed in the Classroom Exercises.

Theorem 4-13

If an angle of a parallelogram is a right angle, then the parallelogram is a rectangle.

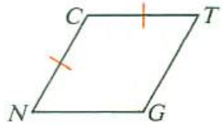
Theorem 4-14

If two consecutive sides of a parallelogram are congruent, then the parallelogram is a rhombus.

Classroom Exercises

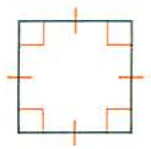
1. When you know that one angle of a parallelogram is a right angle, you can prove that the parallelogram is a rectangle. Explain, using the figure below.

Ex. 1

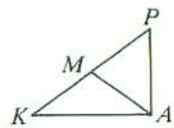


Ex. 2

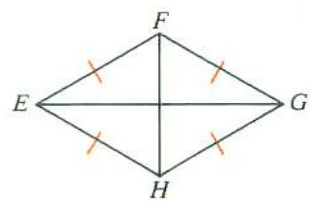
2. When you know that two consecutive sides of a parallelogram are congruent, you can prove that the parallelogram is a rhombus. Explain, using the figure above.
3. Theorem 4–9 can be stated: If a quadrilateral is a rectangle, then the diagonals are congruent. State the converse. Draw a figure to show that the converse is not true.
4. State the converse of Theorem 4–10. Draw a figure to show that the converse is not true.
5. Can you assert that the figure shown is a polygon? a quadrilateral? a parallelogram? a rectangle? a rhombus? a square? Which term describes the figure most effectively?



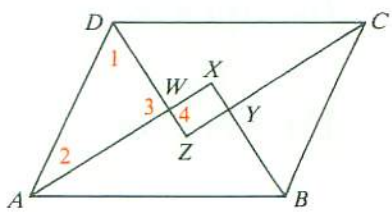
$\angle KAP$ is a right angle, and $\overline{AM}$ is a median.



6. If $MP = 6\frac{1}{2}$, $MA =$ _?_.
7. If $MA = t$, $KP =$ _?_.
8. Given: Rhombus $EFGH$
   a. F, being equidistant from E and G, must lie on the _?_ of $\overline{EG}$.
   b. H, being equidistant from E and G, must lie on the _?_ of $\overline{EG}$.
   c. From (a) and (b) you can deduce that $\overline{FH}$ is the _?_ of $\overline{EG}$.
   d. State the theorem of this section that you have just proved.



9. In the figure, quad. $ABCD$ is a $\square$. $\overrightarrow{AX}$, $\overrightarrow{BX}$, $\overrightarrow{CZ}$, and $\overrightarrow{DZ}$ bisect the angles of the $\square$. Let $m\angle 1 = n$. Then
   $m\angle ADC =$ _?_, $m\angle DAB =$ _?_, $m\angle 2 =$ _?_,
   $m\angle 3 =$ _?_, $m\angle 4 =$ _?_
   You can show, similarly, that the measure of each of the other three angles of quad. $WXYZ$ is _?_. Complete the statement of a theorem we have proved: When the bisectors of the angles of a parallelogram are drawn, _?_.



10. Draw a rectangle and bisect its angles. What name best describes the quadrilateral formed?

Written Exercises

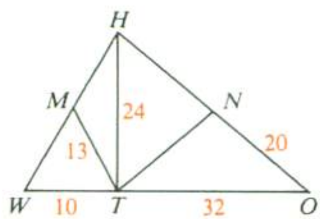
Copy the chart. Then place check marks in the appropriate spaces.

	Property	Parallelogram	Rectangle	Rhombus	Square
A **1.**	Opp. sides are ∥.				
2.	Opp. sides are ≅.				
3.	Opp. ∠s are ≅.				
4.	A diag. forms two ≅ △s.				
5.	Diags. bisect each other.				
6.	Diags. are ≅.				
7.	Diags. are ⊥.				
8.	A diag. bisects two ∠s.				
9.	All ∠s are rt. ∠s.				
10.	All sides are ≅.				

11. Explain why an equiangular quadrilateral must be a rectangle.

12. Explain why a quadrilateral that is a regular polygon must be a square.

$\overline{HT}$ is an altitude of $\triangle HOW$. M and N are the midpoints of $\overline{WH}$ and $\overline{OH}$.



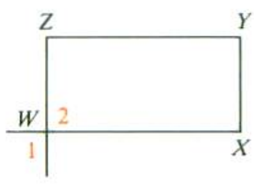
13. $MW = \underline{\ ?\ }$

14. $MH = \underline{\ ?\ }$

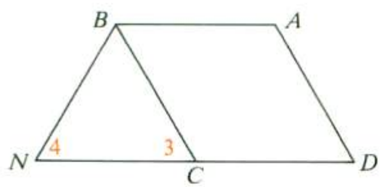
15. $NT = \underline{\ ?\ }$

16. $HO = \underline{\ ?\ }$

17. Given: ▱$WXYZ$; $m\angle 1 = 90$
Prove: $WXYZ$ is a rectangle.

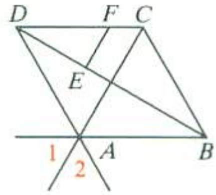


18. Given: ▱$ABCD$; $DC = BN$; $\angle 3 \cong \angle 4$
Prove: $ABCD$ is a rhombus.



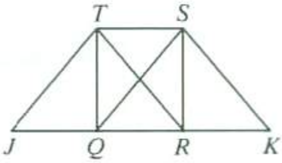
19. Given: Rhombus $ABCD$
Prove: $\angle 1 \cong \angle 2$

20. Given: Rhombus $ABCD$; $\overline{EF} \parallel \overline{AC}$
Prove: $\overline{EF} \perp \overline{DB}$

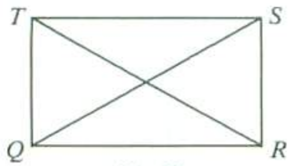


B 21. Given: Rectangle $QRST$; $\square RKST$
Prove: $\triangle QSK$ is isos.

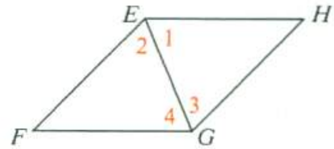
22. Given: Rectangle $QRST$; $\square RKST$; $\square JQST$
Prove: $\overline{JT} \cong \overline{KS}$



23. Using the figure below, write a complete proof of Theorem 4-9. (*Hint:* Prove $\triangle TQR \cong \triangle SRQ$.)



Ex. 23



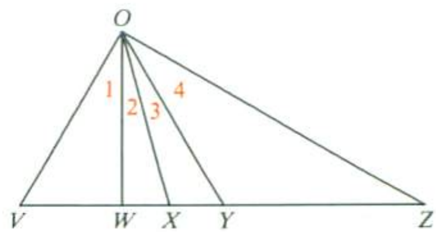
Ex. 24

24. Using the figure above, write a complete proof of Theorem 4-11 for one diagonal of the rhombus. (Note that a proof for the other diagonal would be similar, step-by-step.)

25. Prove: If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

26. Prove: If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

In the figure, $m\angle VOZ = 90$.
$\overline{OW}$ is an altitude of $\triangle VOZ$.
$\overline{OX}$ bisects $\angle VOZ$.
$\overline{OY}$ is a median of $\triangle VOZ$.
Find the measures of the four numbered angles.



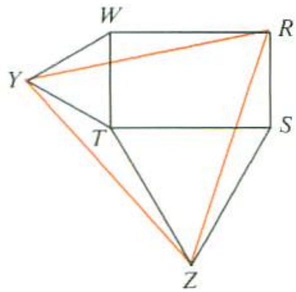
27. $m\angle Z = 30$

28. $m\angle Z = k$

C 29. a. It is known that two sides of a quadrilateral are parallel and that one diagonal bisects an angle. Does that quadrilateral have to be special in other ways? If so, write a proof. If not, draw a convincing figure.
b. Repeat part (a) with stronger conditions: It is known that two sides are parallel and that one diagonal bisects two angles of the quadrilateral.

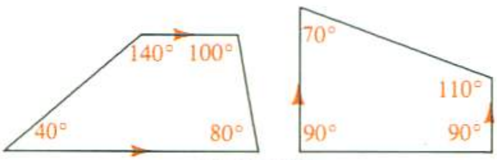
30. Draw a regular pentagon $ABCDE$. Let X be the intersection of $\overline{AC}$ and $\overline{BD}$. What special kind of quadrilateral is $AXDE$? Write a paragraph proof.

31. Given: Rectangle $RSTW$;
equilateral $\triangle$s YWT and STZ
What is true of $\triangle RYZ$?
Write a paragraph proof.

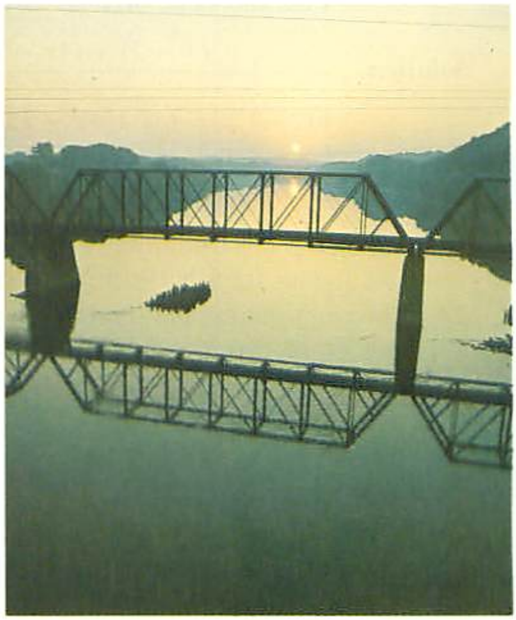


4-4 Trapezoids

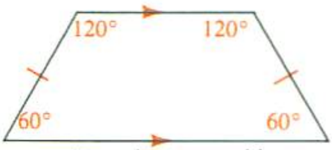
A quadrilateral with exactly one pair of parallel sides is called a **trapezoid.** The parallel sides are called **bases;** the other sides are **legs.**



Trapezoids



A trapezoid with congruent legs is called **isosceles.** As the figure below suggests, both pairs of *base angles* of an isosceles trapezoid are congruent.



Isosceles trapezoid

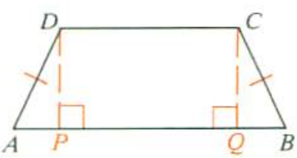
A trapezoidal shape can be seen in the photograph. Is it isosceles?

Theorem 4-15

Base angles of an isosceles trapezoid are congruent.

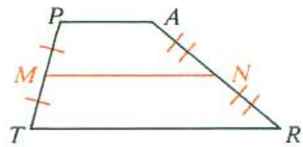
Given: Trapezoid $ABCD$; $\overline{AD} \cong \overline{BC}$

Prove: $\angle A \cong \angle B$; $\angle ADC \cong \angle BCD$



Plan for Proof: Draw segments from D and C perpendicular to $\overleftrightarrow{AB}$. Show that $\overline{DP} \cong \overline{CQ}$, rt. $\triangle APD \cong$ rt. $\triangle BQC$, and $\angle A \cong \angle B$. Since $\angle ADC$ is supp. to $\angle A$, and $\angle BCD$ is supp. to $\angle B$, it follows that $\angle ADC \cong \angle BCD$.

The **median** of a trapezoid is the segment that joins the midpoints of the legs. $\overline{MN}$ is the median of trapezoid $TRAP$.

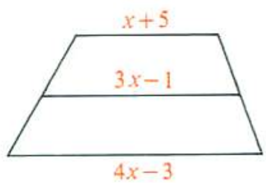


The next theorem could be proved now. Instead you will see a shorter, more appealing proof in Chapter 11.

Theorem 4-16

The median of a trapezoid
(1) is parallel to the bases;
(2) has a length equal to half the sum of the lengths of the bases.

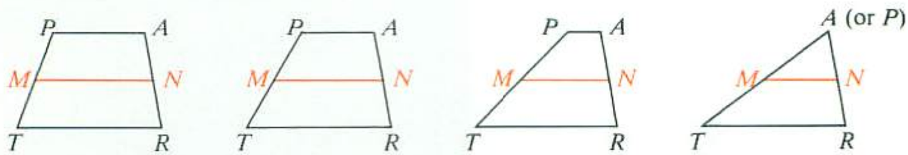
Example A trapezoid and its median are shown. Find the value of x.



Solution

$$3x - 1 = \frac{(x + 5) + (4x - 3)}{2}$$
$$6x - 2 = x + 5 + 4x - 3$$
$$6x - 2 = 5x + 2$$
$$x = 4$$

In the diagrams below, $\overline{MN}$ joins the midpoints of two segments. As you study the figures from left to right, notice that the upper base of trapezoid $TRAP$ becomes shorter and shorter. Finally, $\overline{PA}$ shrinks to a single point and trapezoid $TRAP$ becomes $\triangle TRA$.



If you think of the last figure as a trapezoid with $PA = 0$, then Theorem 4-16 suggests the following:

(1) $\overline{MN} \parallel \overline{TR}$
(2) $MN = \frac{1}{2}(TR + 0) = \frac{1}{2}TR$

Theorem 4-17, which states these properties for $\triangle TRA$, will be proved in Chapter 11.

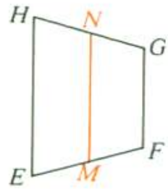
Theorem 4-17

The segment that joins the midpoints of two sides of a triangle
(1) is parallel to the third side;
(2) has a length equal to half the length of the third side.

Classroom Exercises

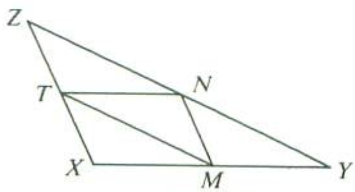
1. In trapezoid $ABCD$, $m\angle A = 70$ and $m\angle C = 120$. Then $m\angle B = \underline{\ ?\ }$ and $m\angle D = \underline{\ ?\ }$.
2. Suppose trapezoid $ABCD$ is isosceles and that $m\angle A = 3j$. Find the measures of $\angle B$, $\angle C$, and $\angle D$ in terms of j.

$\overline{MN}$ is the median of trapezoid *EFGH*.



3. If $HN = 4$ and $EM = 6$, $NG = \underline{\ ?\ }$ and $EF = \underline{\ ?\ }$.
4. If $HE = 16$ and $GF = 10$, $MN = \underline{\ ?\ }$.
5. If $GF = 5$ and $NM = 7$, $HE = \underline{\ ?\ }$.
6. If $HE = 12k$ and $NM = 9k$, $GF = \underline{\ ?\ }$.

M, N, and T are the midpoints of the sides of $\triangle XYZ$.

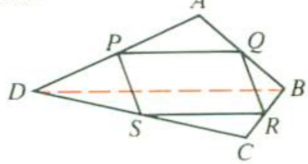


Exs. 7–12

7. If $XZ = 10$, $MN = \underline{\ ?\ }$.
8. If $TN = 7$, $XY = \underline{\ ?\ }$.
9. If $ZN = 8$, $TM = \underline{\ ?\ }$.
10. If $XY = k$, $TN = \underline{\ ?\ }$.
11. Suppose $XY = 10$, $YZ = 14$, and $XZ = 8$. What are the lengths of the three sides of
    a. $\triangle TNZ$? **b.** $\triangle MYN$? **c.** $\triangle XMT$? **d.** $\triangle NTM$?
12. State a theorem suggested by Exercise 11.

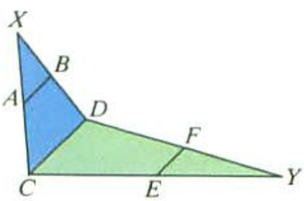
Draw the trapezoid described. If a trapezoid can't be drawn, explain why not.

13. With two right angles
14. With bases shorter than the legs
15. With congruent bases
16. With three acute angles
17. P, Q, R, and S are the midpoints of the sides of quad. $ABCD$.
    a. Explain how diagonal $\overline{BD}$ helps you prove that $\overline{PQ} \cong \overline{SR}$.
    b. How could you prove that $\overline{PQ} \parallel \overline{SR}$?
    c. State the theorem that tells you quad. $PQRS$ is a parallelogram.



Written Exercises

Points A, B, E, and F are the midpoints of $\overline{XC}$, $\overline{XD}$, $\overline{YC}$, and $\overline{YD}$. Complete.

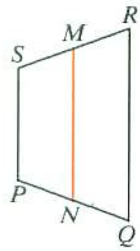


A

1. If $CD = 24$, $AB = \underline{\ ?\ }$ and $EF = \underline{\ ?\ }$.
2. If $AB = k$, $CD = \underline{\ ?\ }$ and $EF = \underline{\ ?\ }$.
3. If $AB = 5x - 8$ and $EF = 3x$, $x = \underline{\ ?\ }$.
4. If $CD = 8x$ and $AB = 3x + 2$, $x = \underline{\ ?\ }$.

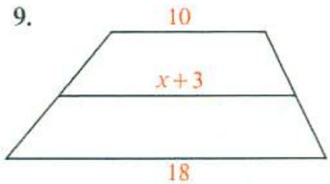
$\overline{MN}$ **is the median of trap.** ***PQRS*. Complete the table.**

	SP	*MN*	*RQ*
5.	9	?	13
6.	12	14	?
7.	3.4	?	5.2
8.	?	$4\frac{1}{2}$	$5\frac{3}{4}$

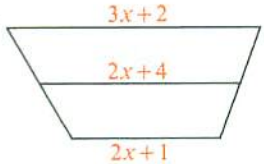


Each figure shows a trapezoid and its median. Find the value of x.

9.



10.



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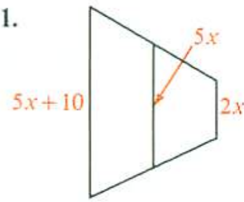
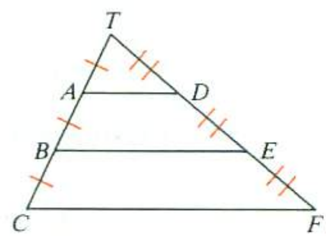


In Exercises 12–16: $TA = AB = BC$ **and** $TD = DE = EF$.



Exs. 12–20

12. Compare lengths AD and BE. (*Hint:* Think of $\triangle TBE$.)

13. Compare the lengths AD, BE, and CF. (*Hint:* Think of trap. $CFDA$.)

14. If $AD = 7$, then $BE =$ _?_ and $CF =$ _?_.

15. If $BE = 26$, then $AD =$ _?_ and $CF =$ _?_.

16. If $AD = x$ and $BE = x + 6$, then $x =$ _?_ and $CF =$ _?_ (numerical answers).

B **17.** If $AD = x + 3$, $BE = x + y$, and $CF = 36$, then $x =$ _?_ and $y =$ _?_.

18. If $AD = x + y$, $BE = 20$, and $CF = 4x - y$, then $CF =$ _?_ (numerical answer).

19. Tony makes up a problem for the figure, setting $AD = 5$ and $CF = 17$. Katie says, "You can't do that." Explain.

20. Mike makes up a problem for the figure, setting $AD = 2x + 1$, $BE = 4x + 2$, and $CF = 6x + 3$ and asking for the value of x. This time Katie says, "Anybody can do that problem." Explain.

Draw a quadrilateral of the type named. Join, in order, the midpoints of the sides. What special kind of quadrilateral do you get?

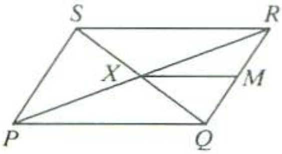
21. Rhombus

22. Rectangle

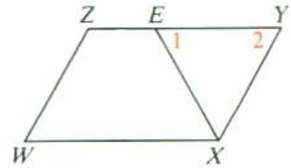
23. Trapezoid

24. Isosceles trapezoid

25. Given: ▱$PQRS$;
M is the midpoint of $\overline{QR}$.
Prove: $MX = \frac{1}{2}PQ$

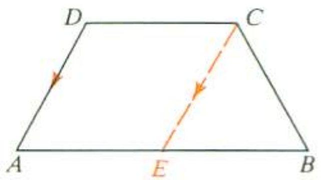


26. Given: ▱$WXYZ$;
$\angle 1 \cong \angle 2$
Prove: $WXEZ$ is an isos. trap.

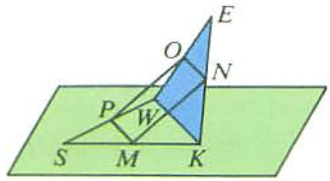


27. Write a proof of Theorem 4-15, following the plan on page 173.

28. Write a proof of Theorem 4-15, using the method suggested by the diagram shown below.



Ex. 28



Ex. 29

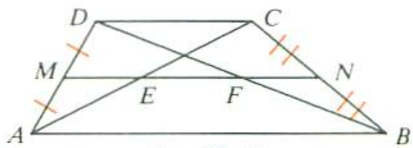
29. A *skew quadrilateral SKEW* is shown. M, N, O, and P are the midpoints of $\overline{SK}$, $\overline{KE}$, $\overline{WE}$, and $\overline{SW}$. Explain why $PMNO$ is a parallelogram.

30. Discover, state, and prove a theorem about the diagonals of an isosceles trapezoid.

31. Prove that a line drawn through the midpoint of one leg of a trapezoid and parallel to the bases bisects the other leg.

32. $DC = 6$ and $AB = 16$.
Find ME, FN, and EF.

C **33.** $DC = 3x$, $AB = 2x^2$, and $EF = 7$.
Find the value of x.

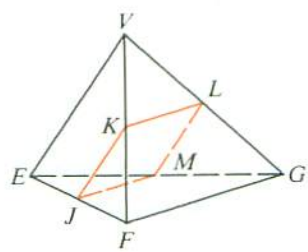


Exs. 32, 33

34. Prove that the perpendicular bisector of one base of an isosceles trapezoid is also the perpendicular bisector of the other base of the trapezoid.

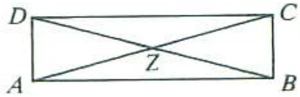
35. State and prove the converse of the theorem you discovered in Exercise 30. (*Hint:* Draw auxiliary lines as in the Plan for Proof for Theorem 4-15 on page 173.)

★ **36.** $\overline{VE}$, $\overline{VF}$, $\overline{VG}$, $\overline{EF}$, $\overline{FG}$, and $\overline{GE}$ are congruent. J, K, L, and M are the midpoints of $\overline{EF}$, $\overline{VF}$, $\overline{VG}$, and $\overline{EG}$. What name best describes $JKLM$? Explain.



Self-Test 1

The diagonals of $\square ABCD$ intersect at Z. Tell whether each statement *must be*, *may be*, or *cannot be* true.

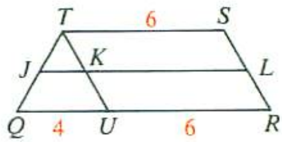


1. $\overline{AC} \cong \overline{BD}$
2. $\overline{DZ} \cong \overline{BZ}$
3. $\overline{AD} \parallel \overline{BC}$
4. $m\angle DAB = 85$ and $m\angle BCD = 95$

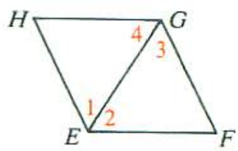
Quad. $WXYZ$ must be a special figure to meet the conditions stated. Write the best name for that special quadrilateral.

5. $\overline{WX} \cong \overline{YZ}$ and $\overline{WX} \parallel \overline{YZ}$
6. $\overline{WX} \parallel \overline{YZ}$ and $\overline{WX} \not\cong \overline{YZ}$
7. $\overline{WX} \cong \overline{YZ}$, $\overline{XY} \cong \overline{ZW}$, and diag. $\overline{WY} \cong$ diag. $\overline{XZ}$
8. Diagonals $\overline{WY}$ and $\overline{XZ}$ are congruent and are perpendicular bisectors of each other.

In the figure, $\overline{TS} \parallel \overline{JL} \parallel \overline{QR}$.



9. Name four trapezoids in the figure.
10. If $\overline{JL}$ is the median of trap. $QRST$, then $JL =$ __?__ and $JK =$ __?__.



11. Given: $\overline{EH} \cong \overline{FG}$; $\overline{EH} \parallel \overline{FG}$
    Prove: $\overline{EF} \cong \overline{HG}$; $\overline{EF} \parallel \overline{HG}$
12. Given: $\angle 1 \cong \angle 2 \cong \angle 3 \cong \angle 4$
    Prove: Quad. $EFGH$ is a rhombus.

B I O G R A P H I C A L   N O T E

Benjamin Banneker

Benjamin Banneker (1731–1806) was a noted American scholar, largely self-taught, who became both a surveyor and an astronomer.

As a surveyor, Banneker was a member of the commission that defined the boundary line and laid out the streets of the District of Columbia.

As an astronomer, he accurately predicted a solar eclipse in 1789. From 1791 until his death he published almanacs containing not only information on astronomy and tide tables, but also such diverse subjects as insect life and medicinal products. Banneker's almanacs included ideas that were far ahead of their time, for example, the formation of a Department of the Interior and even an organization like the United Nations.

Geometric Inequalities

Objectives

1. Write indirect proofs.
2. State and apply the inequality relations for one triangle.
3. State and apply the inequality relations for two triangles.

4-5 Indirect Proofs

Until now the proofs you have written have been *direct* proofs. Sometimes it is difficult or even impossible to find a direct proof, but easy to reason indirectly. In an **indirect proof** you begin by assuming temporarily that the conclusion is not true. Then you reason logically until you reach a contradiction of the hypothesis or another known fact.

Example 1

Given: n is an integer and n^2 is even.

Prove: n is even.

Proof:

Assume temporarily that n is not even. Then n is odd, and

$$\begin{aligned} n^2 &= n \times n \\ &= \text{odd} \times \text{odd} = \text{odd}. \end{aligned}$$

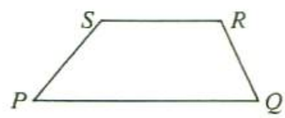
But this contradicts the given information that n^2 is even. Therefore the temporary assumption that n is not even must be false. It follows that n is even.

Example 2

Prove: The bases of a trapezoid have unequal lengths.

Given: Trap. $PQRS$ with bases $\overline{PQ}$ and $\overline{SR}$

Prove: $PQ \neq SR$



Proof:

Assume temporarily that $PQ = SR$. Then $\overline{PQ} \cong \overline{SR}$. Since $\overline{PQ} \parallel \overline{SR}$, by the definition of a trapezoid, quadrilateral $PQRS$ has two sides that are both congruent and parallel. Therefore quadrilateral $PQRS$ must be a parallelogram, and $\overline{PS}$ must be parallel to $\overline{QR}$. But this contradicts the fact that trapezoid $PQRS$ can have only one pair of parallel sides. Our temporary assumption that $PQ = SR$ must be false. It follows that $PQ \neq SR$.

How to Write an Indirect Proof

1. Assume temporarily that the conclusion is not true.
2. Reason logically until you reach a contradiction of a known fact.
3. Point out that your temporary assumption must be false, and that the conclusion must then be true.

Classroom Exercises

1. An indirect proof is to be used to prove: If $AB = AC$, then $\triangle ABD \cong \triangle ACD$. Which one of the following is the correct way to begin?
Assume temporarily that

$$AB \neq AC.$$

Assume temporarily that

$$\triangle ABD \not\cong \triangle ACD.$$

2. Planning to write an indirect proof that $x > 7$, Becky begins by saying, "Assume temporarily that $x < 7$." Using this assumption, she reaches a contradiction. Then she concludes that $x > 7$. Criticize her conclusion.

3. Wishing to prove that l and m are skew lines, John begins an indirect proof by assuming temporarily that l and m are intersecting lines. Using this assumption, he reaches a contradiction and concludes that l and m are skew. Criticize the proof.

What is the first sentence of an indirect proof of the statement shown?

4. $\triangle ABC$ is equilateral.

5. Doug is a Canadian.

6. $a \geq b$

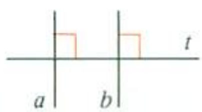
7. Kim isn't a violinist.

8. $m\angle X > m\angle Y$

9. $\overline{CX}$ isn't a median of $\triangle ABC$.

10. Arrange sentences (a)-(e) in an order that completes the indirect proof of the statement: In a plane, two lines perpendicular to a third line are parallel to each other.

Given: Lines a, b, and t lie in a plane;
$t \perp a$; $t \perp b$
Prove: $a \parallel b$



Proof:

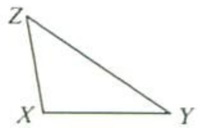
(a) Then a intersects b in some point Z.
(b) But this contradicts Theorem 2-9.
(c) It is false that a is not parallel to b, and it follows that $a \parallel b$.
(d) Assume temporarily that a is not parallel to b.
(e) Then there are two lines through Z and perpendicular to t.

Written Exercises

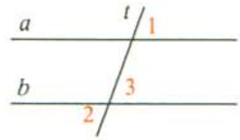
Write indirect proofs in paragraph form.

A

1. Given: $\triangle XYZ$; $m\angle X = 100$
Prove: $\angle Y$ is an acute $\angle$.

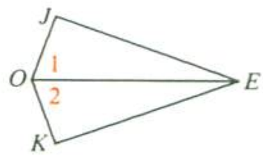


2. Given: Transversal t cuts lines a and b; $m\angle 1 \neq m\angle 2$
Prove: $a \nparallel b$



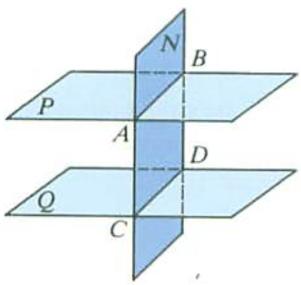
3. Given: $\overline{OJ} \cong \overline{OK}$; $\overline{JE} \not\cong \overline{KE}$
Prove: $\overrightarrow{OE}$ doesn't bisect $\angle JOK$.

4. Given: $\angle 1 \cong \angle 2$; $\overline{OJ} \not\cong \overline{OK}$
Prove: $\angle J$ and $\angle K$ are not both right angles.

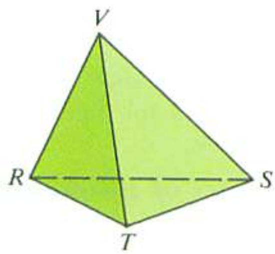


B

5. Given: $\overline{AB} \nparallel \overline{CD}$
Prove: Planes P and Q intersect.

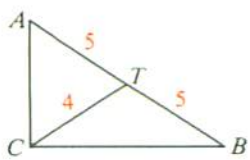


6. Given: $\triangle$s RVT and SVT are equilateral; $\triangle RVS$ is not equilateral.
Prove: $\triangle RST$ is not equilateral.

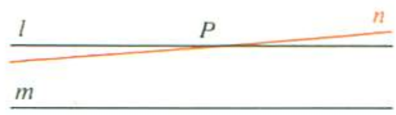


7. Given: Points E, F, G, H; segments $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HE}$;
$m\angle EFG = 93$; $m\angle FGH = 70$; $m\angle GHE = 127$; $m\angle HEF = 60$
Prove: E, F, G, and H are not coplanar.

8. Given: $AT = BT = 5$; $CT = 4$
Prove: $\angle ACB$ is not a rt. $\angle$.



9. Given: Coplanar lines l, m, n; n intersects l in P; $l \parallel m$
Prove: n intersects m.



10. Prove that there is no smallest positive number.

11. Prove that a collection of quarters and dimes worth 95¢ must have an odd number of quarters.

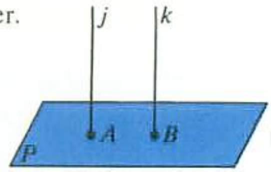
12. Prove that no regular polygon has a 155° angle.

13. Prove that the diagonals of a trapezoid do not bisect each other.

C 14. Prove: If r and s are positive integers, then $r + s \geq \sqrt{r^2 + s^2}$.

15. Prove that if two lines are perpendicular to the same plane, then the lines do not intersect.

16. Given: $\overleftrightarrow{RS}$ and $\overleftrightarrow{TW}$ are skew.
Prove: $\overleftrightarrow{RT}$ and $\overleftrightarrow{SW}$ are skew.

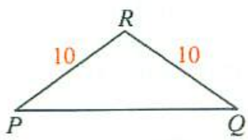


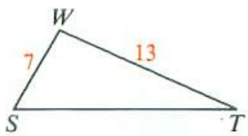
Ex. 15

4-6 Inequalities for One Triangle

From the information given in the figure at the left below you can deduce that $\angle P \cong \angle Q$. From the figure at the right you can deduce that $m\angle S \neq m\angle T$. (If the measures were equal, WS and WT would have to be equal.) The figure suggests a result we will prove in this section:

$$\text{If } TW > SW, \text{ then } m\angle S > m\angle T.$$





Statements such as $m\angle S \neq m\angle T$ and $m\angle S > m\angle T$ are called *inequalities*. Some of the algebraic properties used in dealing with inequalities are listed below.

Properties of Inequalities

If $a > b$ and $c \geq d$, then $a + c > b + d$.

If $a > b$ and $c > 0$, then $ac > bc$.

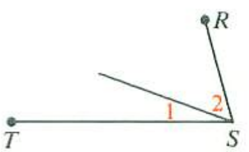
If $a > b$ and $b > c$, then $a > c$.

If $a = b + c$, and $c > 0$, then $a > b$.

Here are examples that show how the last property above can be used.

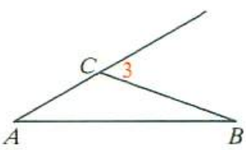
Y is between X and Z.

$XZ = XY + YZ$
Then $XZ > XY$
and $XZ > YZ$.



$\angle 1$ and $\angle 2$ are adj. $\angle$s.

$m\angle RST = m\angle 1 + m\angle 2$
Then $m\angle RST > m\angle 1$
and $m\angle RST > m\angle 2$.



$\angle 3$ is an exterior $\angle$.

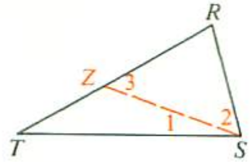
$m\angle 3 = m\angle A + m\angle B$
Then $m\angle 3 > m\angle A$
and $m\angle 3 > m\angle B$.

In a proof you could give as a reason for the final sentence in each example above either "a property of inequalities" or "by algebra."

In the proof of the following theorem, look for two of the situations discussed on page 182.

Theorem 4–18

If one side of a triangle is longer than a second side, then the angle opposite the first side is larger than the angle opposite the second side.



Given: $\triangle RST$; $RT > RS$

Prove: $m\angle RST > m\angle T$

Proof:

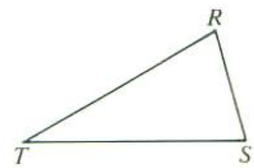
By the Ruler Postulate, there is a point Z on $\overline{RT}$ such that $RZ = RS$. Draw $\overline{SZ}$. In isosceles $\triangle RZS$, $m\angle 3 = m\angle 2$.

Since $m\angle RST > m\angle 2$,
$m\angle RST > m\angle 3$ by substitution.

Since $m\angle 3 > m\angle T$,
$m\angle RST > m\angle T$.

Theorem 4–19

If one angle of a triangle is larger than a second angle, then the side opposite the first angle is longer than the side opposite the second angle.



Given: $\triangle RST$; $m\angle S > m\angle T$

Prove: $RT > RS$

Proof:

Assume temporarily that $RT \not> RS$. Then either $RT = RS$ or $RT < RS$.
If $RT = RS$, $m\angle S = m\angle T$.
If $RT < RS$, $m\angle S < m\angle T$ by Theorem 4–18.
In either case we have a contradiction of the fact that $m\angle S > m\angle T$. The property $RT \not> RS$ that we assumed to be true must be false. It follows that $RT > RS$.

Corollary 1

The perpendicular segment from a point to a line is the shortest segment from the point to the line.

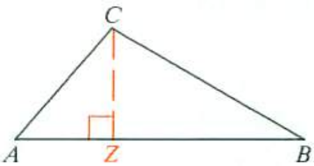
Corollary 2

The perpendicular segment from a point to a plane is the shortest segment from the point to the plane.

See Classroom Exercises 17 and 18 for proofs of the corollaries.

Theorem 4-20 The Triangle Inequality

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



Given: $\triangle ABC$

Prove: (1) $AB + BC > AC$
(2) $AB + AC > BC$
(3) $AC + BC > AB$

Proof:

One of the sides, say $\overline{AB}$, is the longest side. (Or $\overline{AB}$ is at least as long as each of the other sides.) Then the first two statements to be proved are true. To prove (3), draw a line, $\overleftrightarrow{CZ}$, through C and perpendicular to $\overleftrightarrow{AB}$. (Through a point outside a line, there is exactly one line perpendicular to the given line.) By Corollary 1 of Theorem 4-19, $\overline{AZ}$ is the shortest segment from A to $\overline{CZ}$. Also, $\overline{BZ}$ is the shortest segment from B to $\overline{CZ}$. Thus:

$$AC > AZ \quad \text{and} \quad BC > ZB$$
$$AC + BC > AZ + ZB$$
$$AC + BC > AB$$

Example The lengths of two sides of a triangle are 3 and 5. The length of the third side must be greater than __?__, but less than __?__.

Solution Let x be the length of the third side.

$$x + 3 > 5 \qquad 3 + 5 > x \qquad x + 5 > 3$$
$$x > 2 \qquad 8 > x \qquad x > -2$$

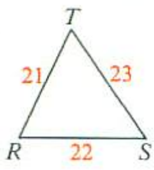
The length of the third side must be greater than 2 but less than 8.

Note that the inequality $x + 5 > 3$ did not give us any useful information. We could have omitted it completely from the solution. (The only inequality that can be omitted is the one in which the length of the third side is added to the longer of the sides with known lengths.)

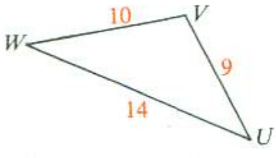
Classroom Exercises

Name the largest angle and the smallest angle of the triangle.

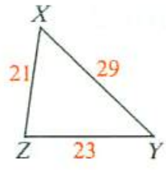
1.



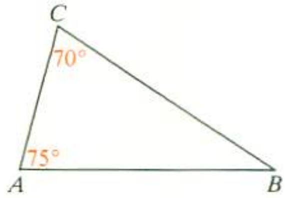
2.

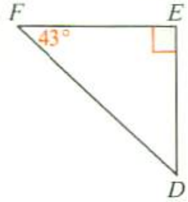


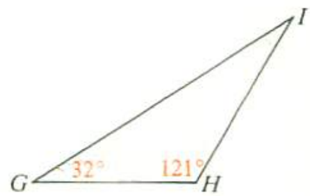
3.



Name the longest side and the shortest side of the triangle.

4. 

5. 

6. 

Is it possible for a triangle to have sides with the lengths indicated?

7. 10, 9, 8 **8.** 6, 6, 20 **9.** 7, 7, 14.1

10. 16, 11, 5 **11.** 0.6, 0.5, 1 **12.** 18, 18, 0.06

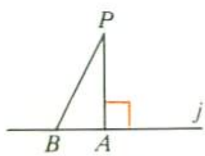
13. The base of an isosceles triangle has length 10. What can you say about the length of a leg?

14. Two sides of a parallelogram have lengths 10 and 12. What can you say about the lengths of the diagonals?

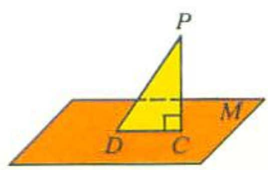
15. Two sides of a triangle have lengths 15 and 20. The length of the third side can be any number between _?_ and _?_.

16. Suppose you know only that the length of one side of a rectangle is 100. What can you say about the length of a diagonal?

17. Use the figure below to explain how Corollary 1 follows from Theorem 4-19.



Ex. 17



Ex. 18

18. Use the figure, in which $\overline{PC} \perp$ plane M, to explain how Corollary 2 follows from Theorem 4-19 or from Corollary 1.

19. Which is the largest angle of a right triangle? Which is the longest side of a right triangle? Explain.

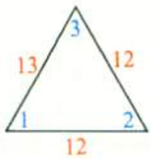
Written Exercises

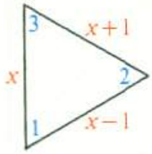
The lengths of two sides of a triangle are given. Write the numbers that best complete the statement: The length of the third side must be greater than _?_, but less than _?_.

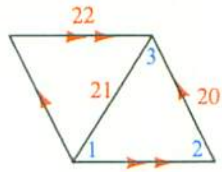
A **1.** 6, 9 **2.** 15, 13 **3.** 100, 100

4. 2.3, 2.3 **5.** k, $k + 5$ **6.** a, b (where $a > b$)

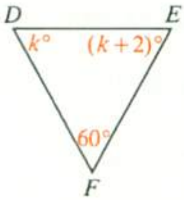
In Exercises 7-9 the diagrams are not drawn to scale. If each diagram were drawn to scale, which of the numbered angles shown would be the largest?

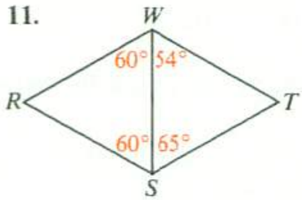
7. 

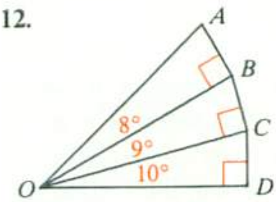
8. 

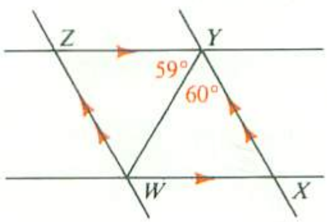
9. 

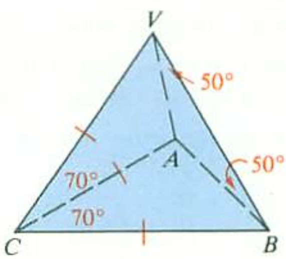
In Exercises 10-14 the diagrams are not drawn to scale. If each diagram were drawn accurately, which segment would be the longest of those shown?

10. 

11. 

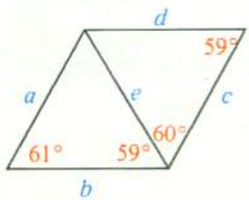
12. 

B 13. 

14. 

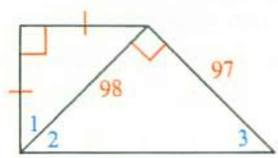
15. Use the lengths a, b, c, d, and e to complete:

___?___ > ___?___ > ___?___ > ___?___ > ___?___



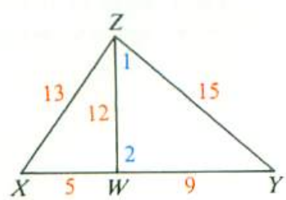
16. Use $m\angle 1$, $m\angle 2$, and $m\angle 3$ to complete:

___?___ > ___?___ > ___?___

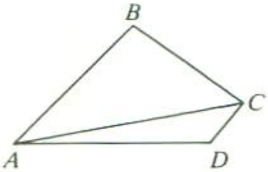


17. The diagram is not drawn to scale. Use $m\angle 1$, $m\angle 2$, $m\angle X$, $m\angle Y$, and $m\angle XZY$ to complete:

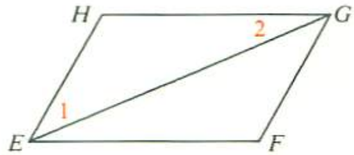
___?___ > ___?___ > ___?___ > ___?___ > ___?___



18. Given: Quad. $ABCD$
Prove: $AB + BC + CD + DA > 2(AC)$



19. Given: $\square EFGH$; $EF > FG$
Prove: $m\angle 1 > m\angle 2$



C 20. Discover, state, and prove in paragraph form a theorem that compares the perimeter of a quadrilateral with the sum of the lengths of the diagonals.

21. Prove that the sum of the lengths of the medians of a triangle is greater than half the perimeter.

22. If you replace "medians" with "altitudes" in Exercise 21, can you prove the resulting statement? Explain.

In Exercises 23 and 24, begin your proofs by drawing auxiliary lines.

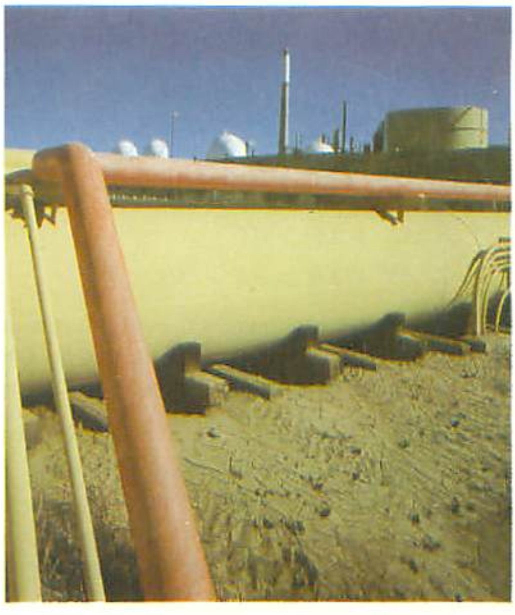
23. Discover, state, and prove a theorem about how the length of the longest side of a quadrilateral compares with the lengths of the other three sides.

24. Prove: If P is any point inside $\triangle XYZ$, then $ZX + ZY > PX + PY$.

Application

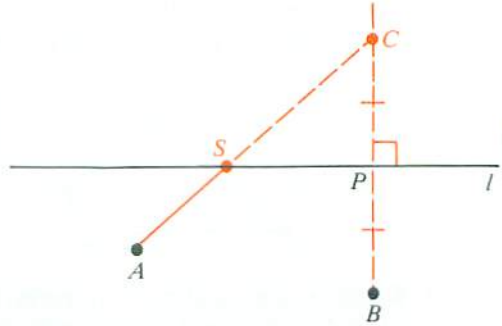
FINDING THE SHORTEST PATH

The owners of pipeline l plan to construct a pumping station at a point S on line l in order to pipe oil to two major customers, located at A and B. To minimize the cost of constructing lines from S to A and B, they wish to locate S along l so that the distance $SA + SB$ is as small as possible.



The construction engineer uses the following method to locate S:

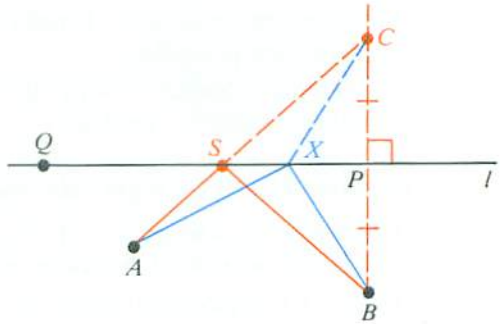
1. Draw a line through B perpendicular to l, intersecting l at point P.
2. On this perpendicular, locate point C so that $PC = PB$.
3. Draw $\overline{AC}$.
4. Locate S at the intersection of $\overline{AC}$ and l.



The diagram shows the path of the new pipelines through the pumping station located at S, and an alternative path going through a different point, X, on l. You can use Theorem 4-20 (the Triangle Inequality) to show that if X is any point on l other than S, then

$$AX + XB > AS + SB.$$

So any alternative path is longer than the path through S.



Exercises

Exercises 1-7 outline the proof that the construction given for S yields the shortest total length for the pipelines serving A and B. Supply a reason for each statement.

1. l is the perpendicular bisector of $\overline{BC}$.

2. $SC = SB$

3. $AS + SC = AC$

4. $AS + SB = AC$

5. $XC = XB$

6. $AX + XC > AC$

7. $AX + XB > AS + SB$

8. Point X is shown between points S and P. However, point X could also be to the right of P or to the left of S. Draw a diagram illustrating each of these cases. Will the same proof still work?

9. The construction for S is sometimes called a *solution by reflection,* since it involves *reflecting* point B in line l. (See Chapter 12 for more about reflections.) Show that $\overline{AS}$ and $\overline{SB}$, like reflected paths of light, make congruent angles with l. That is, prove that $\angle QSA \cong \angle PSB$. (*Hint:* Draw your own diagram, omitting the part of the diagram shown in blue.)

4-7 Inequalities for Two Triangles

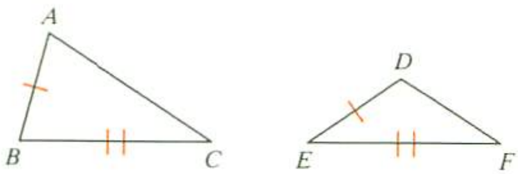
Begin with two matched pairs of sticks joined loosely at B and E. Open them so that $m\angle B > m\angle E$ and you will find that $AC > DF$. Conversely, if you open them so that $AC > DF$, it will follow that $m\angle B > m\angle E$. Two theorems, one of which is surprisingly difficult to prove, are suggested by these examples.

Theorem 4-21 SAS Inequality Theorem

If two sides of one triangle are congruent to two sides of another triangle, but the included angle of the first triangle is greater than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle.

Given: $\overline{BA} \cong \overline{ED}$; $\overline{BC} \cong \overline{EF}$;
$m\angle B > m\angle E$

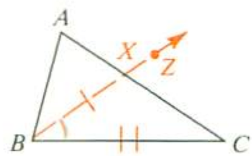
Prove: $AC > DF$



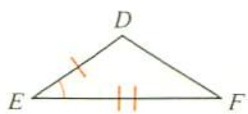
Outline of Proof:

Draw $\overrightarrow{BZ}$ so that $m\angle ZBC = m\angle E$. On $\overrightarrow{BZ}$ take point X so that $BX = ED$.
Then either X is on $\overline{AC}$ or X is not on $\overline{AC}$.
In either case, $\triangle XBC \cong \triangle DEF$ by SAS, and $XC = DF$.

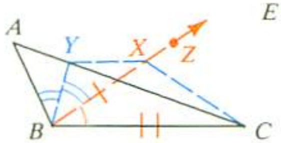
Case 1: X is on $\overline{AC}$.
$AC > XC$ (Seg. Add. Post. and algebra)
$AC > DF$ (Substitution Property, using the equation in red above)



Case 1



Case 2: X is not on $\overline{AC}$.
Draw the bisector of $\angle ABX$, intersecting $\overline{AC}$ at Y.
Draw $\overline{XY}$ and $\overline{XC}$.
$BA = ED = BX$ (Why?)
Since $\triangle ABY \cong \triangle XBY$ (SAS), $AY = XY$.
$XY + YC > XC$ (Why?)
$AY + YC > XC$ (Why?), or $AC > XC$
$AC > DF$ (Substitution Property)



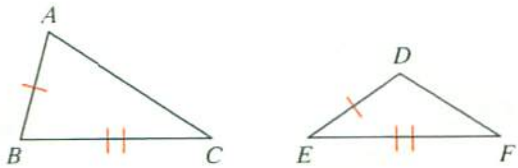
Case 2

Theorem 4-22 SSS Inequality Theorem

If two sides of one triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second, then the included angle of the first triangle is larger than the included angle of the second.

Given: $\overline{BA} \cong \overline{ED}$; $\overline{BC} \cong \overline{EF}$;
$AC > DF$

Prove: $m\angle B > m\angle E$



Proof:

Assume temporarily that $m\angle B \not> m\angle E$.
Then either $m\angle B = m\angle E$ or $m\angle B < m\angle E$.

Case 1: If $m\angle B = m\angle E$, then $\triangle ABC \cong \triangle DEF$ by the SAS Postulate, and $AC = DF$.

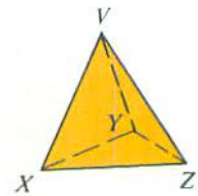
Case 2: If $m\angle B < m\angle E$, then $AC < DF$ by the SAS Inequality Theorem.

In either case we have a contradiction of the fact that $AC > DF$. What we temporarily assumed to be true, that $m\angle B \not> m\angle E$, must be false. It follows that $m\angle B > m\angle E$.

Example What can you deduce from the given information?

a. $VX = VZ$ and $XY > ZY$

b. $VZ = XZ$ and $m\angle XZY > m\angle VZY$

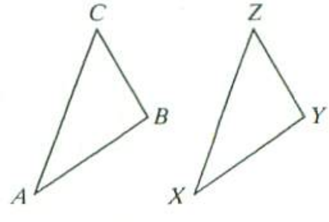


Solution **a.** Apply the SSS Inequality Theorem to $\triangle XVY$ and $\triangle ZVY$ to get $m\angle XVY > m\angle ZVY$.
Use the fact that $XY > ZY$ in $\triangle XYZ$ to get $m\angle YZX > m\angle YXZ$.

b. Apply the SAS Inequality Theorem to $\triangle XZY$ and $\triangle VZY$ to get $XY > VY$.
Apply the fact that $XY > VY$ in $\triangle XYV$ to get $m\angle XVY > m\angle VXY$.

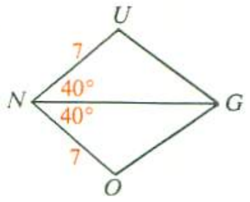
Classroom Exercises

In Exercises 1-10, some facts are given. What can you deduce?



1. $AB = XY$; $AC = XZ$; $m\angle A > m\angle X$
2. $CA = ZX$; $CB = ZY$; $m\angle C < m\angle Z$
3. $BA = YX$; $BC = YZ$; $AC > XZ$
4. $AB = XY$; $AC = XZ$; $\angle B \cong \angle Y$, $m\angle C < m\angle Z$
5. $AB < AC$
6. $m\angle Y > m\angle Z$

7.



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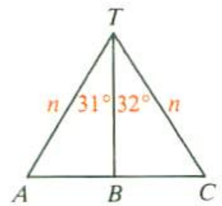

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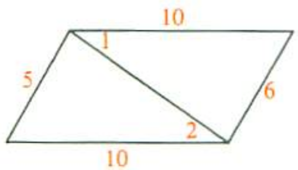

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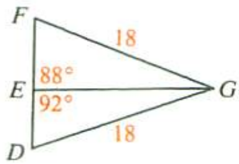
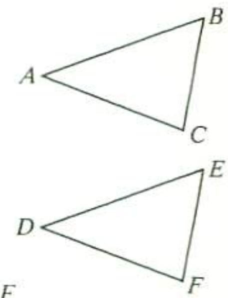


Written Exercises

Choose from the words *always, sometimes,* and *never* to complete the sentence in the best way.



A **1.** If $AB = DE$, $BC = EF$, and $AC = DF$, then $m\angle B$ is _?_ equal to $m\angle E$.

2. If $m\angle B > m\angle A$, then AB is _?_ greater than AC.

3. If $m\angle A > m\angle D$ and $m\angle C > m\angle F$, then $m\angle B$ is _?_ greater than $m\angle E$.

4. If $AB > DE$ and $AC > DF$, then BC is _?_ greater than EF.

5. If $AC = DF$, $AB = DE$, and $CB > FE$, then $m\angle A$ is _?_ greater than $m\angle D$.

6. If $BA = ED$, $BC = EF$, and $m\angle B < m\angle E$, then AC is _?_ less than DF.

7. If $m\angle A > m\angle D$ and $m\angle C > m\angle F$, then AC is _?_ less than DF.

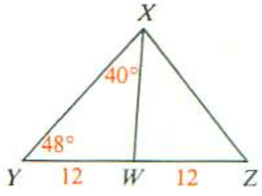
8. If $m\angle A = m\angle D$, $m\angle C < m\angle F$, $BA = ED$, and $BC = EF$, then AC is _?_ less than DF.

9. If $AB > DE$, $CA = FD$, and $CB = FE$, then $m\angle C$ is _?_ greater than $m\angle F$.

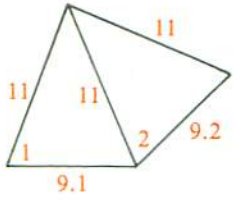
10. If $AC = AB = DF = DE$ and $m\angle C > m\angle F$, then CB is _?_ greater than FE.

11. In the figure below, which is longer, $\overline{XY}$ or $\overline{XZ}$?

12. Which is larger, $\angle 1$ or $\angle 2$?



Ex. 11

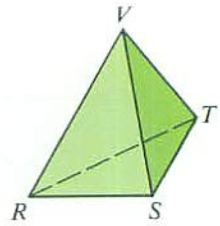


Ex. 12

B **13.** If $RT = RS$ and $VT > VS$, which angle is larger, $\angle VRT$ or $\angle VRS$?

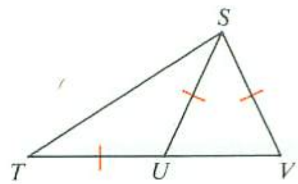
14. If $VR = VS = VT$ and $m\angle RVS > m\angle RVT > m\angle SVT$, which angle of $\triangle RST$ is the largest one?

15. If $TS = TR$ and $m\angle VTS > m\angle VTR$, which angle is larger, $\angle VSR$ or $\angle VRS$?

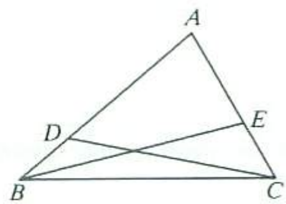


Exs. 13–15

16. Given: $TU = US = SV$
Prove: $ST > SV$



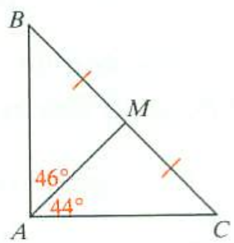
17. Given: $AB > AC$; $BD = EC$
Prove: $BE > CD$



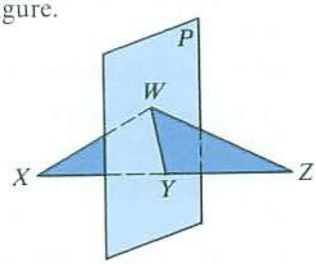
18. Prove: In a parallelogram, the diagonal that joins the vertices of the smaller angles is the longer diagonal.

In Exercises 19–22 write paragraph proofs.

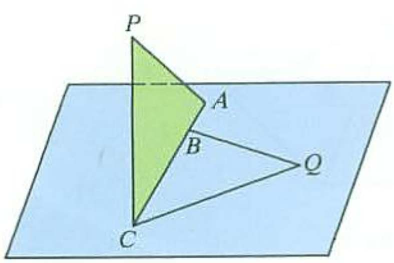
19. Given: M is the midpoint of $\overline{BC}$.
Discover and prove something about BA and CA.



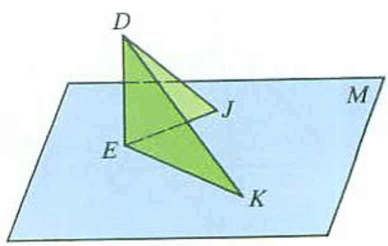
20. Given: Plane P bisects $\overline{XZ}$ at Y; $WZ > WX$.
Discover and prove something about the figure.



C **21.** Given: $PA = PC = QC = QB$
Prove: $m\angle PCA < m\angle QCB$



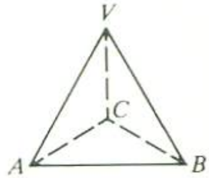
22. Given: $\overline{DE} \perp$ Plane M; $EK > EJ$
Prove: $DK > DJ$
(*Hint:* On $\overline{EK}$ take Z so that $EZ = EJ$.)



23. In the solid shown, five edges have equal lengths, but $\overline{VC}$ has a different length. What can you say about the largest angles of the twelve angles shown if

a. $\overline{VC}$ is the longest edge?

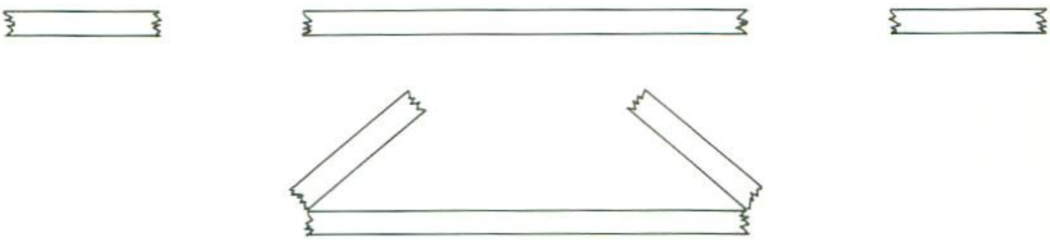
b. $\overline{VC}$ is the shortest edge?



COMPUTER KEY-IN

If you break a stick into three pieces, do you think it is always possible to join the pieces end-to-end to form a triangle?

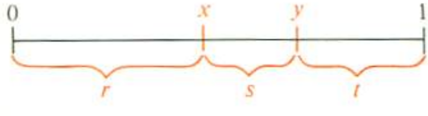
It's easy to see that if the sum of the lengths of any two of the pieces is less than or equal to that of the third, a triangle can't be formed.



By an experiment, your class can estimate the probability that three pieces of broken stick will form a triangle. Suppose everyone in your class has a stick 1 unit long and breaks it into three pieces. If there are thirty people in your class and eight people are able to form a triangle with their pieces, we estimate that the probability of forming a triangle is about $\frac{8}{30}$.

Of course, this experiment is not very practical. You can get much better results by having a computer simulate the breaking of many, many sticks, as in the program in BASIC on the next page.

In lines 30 and 40 of the following program, you tell the computer how many sticks you want to break. Each stick is 1 unit long, and the computer breaks each stick by choosing two random numbers x and y between 0 and 1. These numbers divide the stick into three lengths r, s, and t.



The computer then keeps count of the number of sticks (N) which form a triangle when broken.

Notice that RND is used in lines 70 and 80. Since usage of RND varies, check this with the manual for your computer and make any necessary changes. The computer print-outs shown in this text use capital letters. The x, y, r, s, and t used in the discussion above appear as X, Y, R, S, T.

```
10 PRINT "SIMULATION--BREAKING STICKS TO MAKE TRIANGLES"
20 PRINT
30 PRINT "HOW MANY STICKS DO YOU WANT TO BREAK";
40 INPUT D
50 LET N = 0
60 FOR I = 1 TO D
70 LET X = RND(1)
80 LET Y = RND(1)
90 IF X > Y THEN 120
100 LET R = X
110 GOTO 130
120 LET R = Y
130 LET S = ABS(X - Y)
140 LET T = 1 - R - S
150 IF R + S <= T THEN 210
160 IF S + T <= R THEN 210
170 IF T + R <= S THEN 210
180 PRINT
190 PRINT R,S,T
200 LET N = N + 1
210 NEXT I
220 LET P = N/D
230 PRINT
240 PRINT "THE EXPERIMENTAL PROBABILITY THAT"
250 PRINT "A BROKEN STICK CAN MAKE A TRIANGLE IS";P
260 END
```

Line Number	Explanation
60-140	These lines simulate the breaking of each stick. When I = 10, for example, the computer is "breaking" the tenth stick.
150-170	Here the computer tests to see whether the pieces of the broken stick can form a triangle. If not, the computer goes on to the next stick (line 210) and the value of N is not affected.
200	If the broken stick has survived the tests of steps 150-170, then the pieces can form a triangle and the value of N is increased by 1 here.
210	Lines 60-210 form a loop that is repeated D times. When I = D, the probability P is calculated and printed (lines 220-250).

Exercises

1. Pick any two numbers x and y between 0 and 1 with $x < y$. With paper and pencil, carry out the instructions in lines 90 through 170 of the program to see how the computer finds r, s, and t and tests to see whether the values can be the lengths of the sides of a triangle. Do the same for a pair x and y with $x > y$.

2. If your computer uses a language other than BASIC, write a similar program for your computer.

3. Run the program several times for $D = 40$.

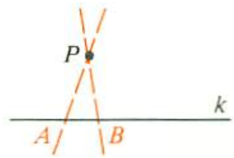
4. Delete the print statements in lines 180 and 190 and then run the program for large values of D, say 100, 400, 800, and compare your results with those of some classmates. Does the probability that the pieces of a broken stick form a triangle appear to be less than or greater than $\frac{1}{2}$?

Self-Test 2

1. Write the letters (a) to (d) in such an order that the sentences provide an indirect proof of the statement: Through a point outside a line, there is at most one line perpendicular to the given line.

   Given: Point P not on line k

   Prove: There is at most one line through P perpendicular to k.



   (a) But this contradicts Corollary 3 of Theorem 2-11: In a triangle, there can be at most one right angle or obtuse angle.

   (b) Then $\angle PAB$ and $\angle PBA$ are right angles, and $\triangle PAB$ has two right angles.

   (c) Thus our temporary assumption is false, and there is at most one line through P perpendicular to k.

   (d) Assume temporarily that there are two lines through P and perpendicular to k at A and B.

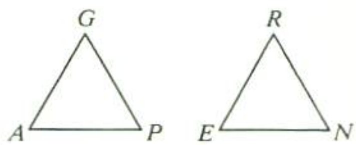
Write the correct symbol ($<$, $=$, $>$) to complete the statement.

2. If $ER > EN$, then $m\angle R$ _?_ $m\angle N$.

3. If $\overline{AG} \cong \overline{ER}$, $\overline{AP} \cong \overline{EN}$, and $\angle A \cong \angle E$, then GP _?_ RN.

4. If $\overline{GA} \cong \overline{RE}$, $\overline{GP} \cong \overline{RN}$, and $AP > EN$, then $m\angle G$ _?_ $m\angle R$.

5. If $PG = NR$, $PA = NE$, and $m\angle P < m\angle N$, then GA _?_ RE.



Exs. 2–5

6. The lengths of the sides of a triangle are 5, 6, and x. Then _?_ $< x <$ _?_.

7. In $\triangle DOM$, $\angle O$ is a right angle and $m\angle D > m\angle M$. Which side of $\triangle DOM$ is the shortest side?

The longer diagonal of $\square QRST$ is $\overline{QS}$. Tell whether each statement *must be*, *may be*, or *cannot be* true.

8. $\angle R$ is an acute angle

9. $QS > RS$

10. $RS > RT$

EXTRA

Non-Euclidean Geometries

When you develop a geometry, you have some choice as to which statements you are going to postulate and which you are going to prove. For example, consider these two statements:

(A) If two parallel lines are cut by a transversal, then corresponding angles are congruent.

(B) Through a point outside a line, there is exactly one line parallel to the given line.

In this book, statement (A) is Postulate 10 and statement (B) is Theorem 2-8. In some books, statement (B) is a postulate and statement (A) is a theorem. In still other developments, both of these statements are proved on the basis of some third statement chosen as a postulate.

A natural question to raise is, Do we have to assume either statement, or an equivalent, at all? Couldn't each of them be proved on the basis of the other postulates? The answer, which wasn't known until the nineteenth century, is that some assumption about parallels *must* be made.

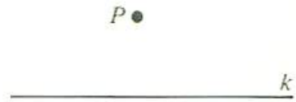
A geometry that provides for a unique parallel to a line through a point not on the line is called *Euclidean,* so this book is a book on Euclidean geometry. Geometries that do not provide for a unique parallel are called *non-Euclidean.* Such geometries, developed in the nineteenth century, aren't mere curiosities; for example, Einstein's Theory of Relativity is based on non-Euclidean geometry. The statements below show the key differences between Euclidean geometry and two types of non-Euclidean geometry.

Euclidean geometry	Through a point outside a line, there is *exactly one* line parallel to the given line.
Hyperbolic geometry	Through a point outside a line, there is *more than one* line parallel to the given line. (Bolyai, Lobachevsky, Gauss)
Elliptic geometry	Through a point outside a line, there is *no* line parallel to the given line. (Riemann)

To see a model of a no-parallel geometry, visualize the surface of a sphere. Think of a *line* as being a great circle of the sphere, that is, the intersection of the sphere and a plane that passes through the center of the sphere. On the sphere, through a point outside a line, there isn't any line parallel to the line. All lines, as defined, intersect.

Any development of Euclidean geometry includes an assumption about parallels equivalent to statement (B). For historical reasons statement (B) is often called *the Parallel Postulate,* but it is only one of many possible choices. In this book we state our assumptions about parallels in Postulates 10 and 11 and prove statement (B) as a theorem. Some advantages of this choice are pointed out below.

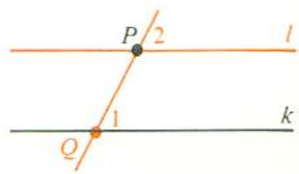
First we show how statement (B) follows from our postulates. Notice that Postulates 10 and 11 play a crucial role in the proof below. In fact, without such assumptions about parallels there couldn't be a proof. Before the discovery of non-Euclidean geometries people didn't know that this was the case and tried, without success, to find a proof that was independent of any assumption about parallels.



Given: Point P outside line k

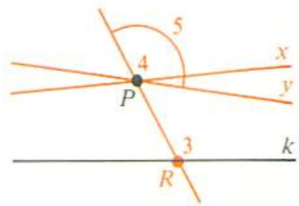
Prove: (1) There is a line through P parallel to k.
(2) There is only one line through P parallel to k.

Outline of proof of (1):



1. Draw a line through P and some point Q on k. (Postulates 5 and 6)
2. Draw line l so that $\angle 2$ and $\angle 1$ are corresponding angles and $m\angle 2 = m\angle 1$. (Protractor Postulate)
3. $l \parallel k$, so there is a line through P parallel to k. (Postulate 11)

Indirect proof of (2):



Assume temporarily that there are at least two lines, x and y, through P parallel to k. Draw a line through P and some point R on k. $\angle 4 \cong \angle 3$ and $\angle 5 \cong \angle 3$ by Postulate 10, so $\angle 5 \cong \angle 4$. But since x and y are different lines we also have $m\angle 5 > m\angle 4$. This is impossible, so our assumption must be false, and it follows that there is only one line through P parallel to k.

This theorem, which is our Theorem 2-8, can be used immediately to prove that the sum of the measures of the angles of a triangle is 180 (Theorem 2-11). Thus our choice of postulates allows us to establish basic facts about angles of polygons near the beginning of our study. When statement (B) is taken as a postulate, the most straightforward proof of statement (A) depends on the SAS Congruence Postulate, so that these facts must wait until triangle congruence has been developed.

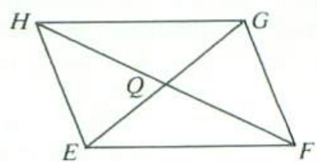
The original parallel postulate of Euclidean geometry is Postulate V in Euclid's *Elements*. A look at the wording of this postulate, which differs from both statement (A) and statement (B), suggests another advantage of our approach to parallels. It reads as follows: "If a straight line meets two other straight lines so as to make the two interior angles on one side of it together less than two right angles, the other straight lines, if extended indefinitely, will meet on that side on which the angles are less than two right angles." Our Postulate 10 is easier to state and makes a more convenient tool for building proofs.

Chapter Summary

1. A parallelogram has these properties:
   a. Opposite sides are parallel.
   b. Opposite sides are congruent.
   c. Opposite angles are congruent.
   d. Diagonals bisect each other.
2. The chart on page 164 lists five ways to prove that a quadrilateral is a parallelogram.
3. If three parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal.
4. A line that contains the midpoint of one side of a triangle and is parallel to another side bisects the third side.
5. Rectangles, rhombuses, and squares are parallelograms with additional properties.
6. The median of a trapezoid is parallel to the bases and has a length equal to half the sum of the lengths of the bases.
7. The segment that joins the midpoints of two sides of a triangle is parallel to the third side and has a length equal to half the length of the third side.
8. You begin an indirect proof by assuming temporarily that what you wish to prove true is *not* true. If this temporary assumption leads to a contradiction of a known fact, then your temporary assumption must be false and what you wish to prove true must be true.
9. In $\triangle RST$, if $RT > RS$, then $m\angle S > m\angle T$. If $m\angle S > m\angle T$, then $RT > RS$.
10. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
11. You can use the SAS and the SSS Inequality Theorems to compare lengths of sides and measures of angles in two triangles.

Chapter Review

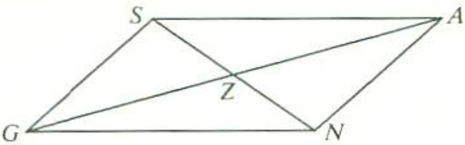
In parallelogram *EFGH*, $m\angle EFG = 70$.



4-1

1. $m\angle HEF =$ __?__
2. If $HQ = 14$, then $HF =$ __?__.
3. If $m\angle EFH = 32$, then $m\angle EHF =$ __?__.
4. If $EH = 8x - 7$ and $FG = 5x + 11$, then $x =$ __?__.

In each exercise you could prove that quad. *SANG* is a parallelogram if one more fact, in addition to those stated, were given. State that fact.



5. $GN = 9$; $NA = 5$; $SA = 9$ 4-2

6. $\angle ASG \cong \angle GNA$

7. $\overline{SZ} \cong \overline{NZ}$

8. $\overline{SA} \parallel \overline{GN}$; $SA = 17$

Write the best name for the figure described.

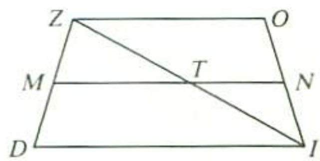
9. A quadrilateral with diagonals that bisect each other. 4-3

10. A rhombus with a right angle.

11. A quadrilateral in which four sides are congruent but not all four angles are congruent.

12. A parallelogram in which two consecutive angles are congruent.

$\overline{MN}$ is the median of trapezoid *ZOID*.



13. The bases of trap. *ZOID* are _?_ and _?_. 4-4

14. If $ZO = 8$ and $MN = 11$, then $DI =$ _?_.

15. If $ZO = 8$, then $TN =$ _?_.

16. If trap. *ZOID* is isosceles and $m\angle D = 80$, then $m\angle O =$ _?_.

17. Write the letters (a) to (d) in such an order that the sentences provide an indirect proof of the statement: If $n^2 + 6 = 32$, then $n \neq 5$. 4-5

(a) But this contradicts the fact that $n^2 + 6 = 32$.

(b) Our temporary assumption must be false, and it follows that $n \neq 5$.

(c) Assume temporarily that $n = 5$.

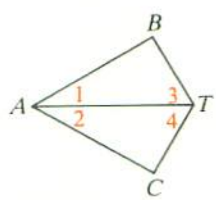
(d) Then $n^2 + 6 = 31$.

18. In $\triangle TEX$, if $TE > XE$, then $m\angle T < m\angle$ _?_. 4-6

19. In $\triangle BAN$, if $m\angle A > m\angle N$, then _?_ $>$ _?_.

20. Two sides of a triangle have lengths 9 and 12. The length of the third side must be greater than _?_ and less than _?_.

Use one of the symbols $<$, $=$, or $>$ to complete the statement.



21. If $\overline{AB} \cong \overline{AC}$ and $m\angle 1 > m\angle 2$, then BT _?_ CT. 4-7

22. If $\overline{TB} \cong \overline{TC}$ and $AB < CA$, then $m\angle 3$ _?_ $m\angle 4$.

23. If $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$, then AB _?_ AC.

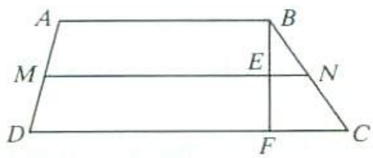
24. If $\overline{TB} \cong \overline{TC}$ and $m\angle 3 > m\angle 4$, then AB _?_ AC.

Chapter Test

Tell whether the statement is *always, sometimes,* or *never* true.

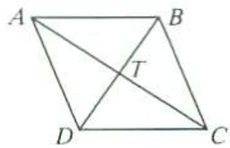
1. A square is ___?___ a rectangle.
2. A rectangle is ___?___ a rhombus.
3. A rhombus is ___?___ a square.
4. A rhombus is ___?___ a parallelogram.
5. A trapezoid ___?___ has three congruent sides.
6. The diagonals of a trapezoid ___?___ bisect each other.
7. The sides of a triangle are ___?___ 13 cm, 19 cm, and 33 cm long.
8. In $\square ABCD$, if $m\angle A > m\angle B$, then $\angle D$ is ___?___ an acute angle.

Trapezoid $ABCD$ has median $\overline{MN}$.

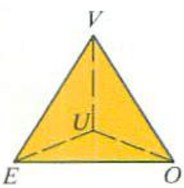


9. If $DC = 17$ and $MN = 12$, then $AB =$ ___?___.
10. If $FC = 9$, then $EN =$ ___?___.
11. If $AB = 5j + 7k$ and $DC = 9j - 3k$, then $MN =$ ___?___.
12. If the sides of a triangle have lengths x, 8, and 12, then ___?___ $< x <$ ___?___.
13. To write an indirect proof of "If $RS = 10$, then quad. $RSTU$ is a parallelogram," you begin by writing: "Assume temporarily that ___?___."

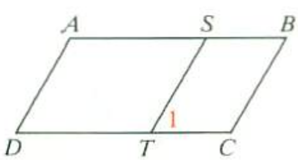
State the theorem that enables you to deduce, from the information given, that quad. $ABCD$ is a parallelogram.



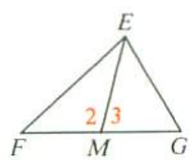
14. $\angle ADC \cong \angle CBA$ and $\angle BAD \cong \angle DCB$
15. $\overline{AD} \parallel \overline{BC}$ and $\overline{AD} \cong \overline{BC}$
16. $AT = CT$ and $DT = \frac{1}{2}DB$
17. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are all congruent.



18. If $VE > VU$, then $m\angle$ ___?___ $> m\angle$ ___?___.
19. If $m\angle EOU > m\angle EUO$, then ___?___ $>$ ___?___.
20. If $\overline{VE} \cong \overline{VO}$ and $m\angle UVE > m\angle UVO$, then ___?___ $>$ ___?___.
21. If $m\angle EVU = 60$, $\overline{OE} \cong \overline{OU}$, and $m\angle VOE > m\angle VOU$, then the largest angle of $\triangle UVE$ is $\angle$ ___?___.
22. Given: $\square ABCD$; $\angle D \cong \angle 1$
    Prove: Quad. $ASTD$ is a parallelogram.



23. Given: $\overline{EM}$ is a median of $\triangle EFG$; $m\angle 2 > m\angle 3$
    Prove: $m\angle G > m\angle F$



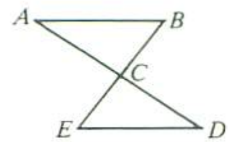
Preparing for College Entrance Exams

Strategy for Success

You may find it helpful to sketch figures or do calculations in your test booklet. Be careful not to make extra marks on your answer sheet.

Indicate the best answer by writing the appropriate letter.

1. Given: $\triangle RGA$ and $\triangle PMC$ with $\overline{RG} \cong \overline{PM}$, $\overline{RA} \cong \overline{PC}$, and $\angle R \cong \angle P$. Which method could be used to prove that $\triangle RGA \cong \triangle PMC$?
(A) SSS **(B)** SAS **(C)** HL **(D)** ASA
(E) There is not enough information for a proof.



2. Given: $\overline{BE}$ bisects $\overline{AD}$. To prove that the triangles are congruent by the AAS method, you must show that:
(A) $\angle A \cong \angle E$ **(B)** $\angle A \cong \angle D$ **(C)** $\angle B \cong \angle E$
(D) $\angle B \cong \angle D$ **(E)** $\angle \overline{AD}$ bisects $\overline{BE}$.

3. Which statement does *not* guarantee that quadrilateral $WXYZ$ is a parallelogram?
(A) $\overline{WX} \cong \overline{YZ}$; $\overline{XY} \parallel \overline{WZ}$ **(B)** $\angle W \cong \angle Y$; $\angle X \cong \angle Z$
(C) $\overline{WX} \cong \overline{YZ}$; $\overline{XY} \cong \overline{WZ}$ **(D)** $\overline{XY} \parallel \overline{WZ}$; $\overline{WX} \parallel \overline{ZY}$
(E) $\overline{XY} \cong \overline{WZ}$; $\overline{XY} \parallel \overline{WZ}$

4. In $\triangle ABC$, $AB = 7$ and $BC = 10$. AC *cannot* equal:
(A) 7 **(B)** 10 **(C)** 3.14 **(D)** 17 **(E)** $\frac{34}{3}$

5. The next number in the sequence 2, 6, 12, 20, 30, 42, _?_ is:
(A) 56 **(B)** 52 **(C)** 58 **(D)** 54 **(E)** 60

6. Which statement is not always true for every rhombus $ABCD$?
(A) $AB = BC$ **(B)** $AC = BD$ **(C)** $\angle B \cong \angle D$ **(D)** $\overline{AC} \perp \overline{BD}$ **(E)** $\angle ABD \cong \angle CBD$

7. The diagonals of quadrilateral $MNOP$ intersect at X. Which statement guarantees that $MNOP$ is a rectangle?
(A) $MO = NP$ **(B)** $\angle PMN \cong \angle MNO \cong \angle NOP$
(C) $MX = NX = OX = PX$ **(D)** $\overline{MO} \perp \overline{NP}$
(E) Each pair of consecutive angles is supplementary.

8. In $\triangle JKL$, $\overline{KL} \cong \overline{LJ}$, $m\angle K = 2x - 36$, and $m\angle L = x + 2$. Find $m\angle J$.
(A) 50 **(B)** 52 **(C)** 53 **(D)** 55 **(E)** 64

9. In $\triangle RST$, $\overleftrightarrow{SU}$ is the perpendicular bisector of $\overline{RT}$ and U lies on $\overline{RT}$. Which statement(s) must be true?
(I) $\triangle RST$ is equilateral (II) $\triangle RSU \cong \triangle TSU$
(III) $\overrightarrow{SU}$ is the bisector of $\angle RST$
(A) I only **(B)** II only **(C)** III only
(D) II and III only **(E)** I, II, and III

10. In $\triangle ABC$, if $AB = BC$ and $AC > BC$, then:
(A) $AB < AC - BC$ **(B)** $m\angle B > m\angle C$ **(C)** $m\angle B < m\angle A$
(D) $m\angle B = 60$ **(E)** $m\angle B = m\angle A$

Cumulative Review: Chapters 1–4

A 1. On a number line, point A has coordinate -5 and point B has coordinate 3. Find the coordinate of the midpoint of $\overline{AB}$.

2. Name the property that justifies the statement:
   a. If $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$, then $\angle 1 \cong \angle 3$.
   b. If $x = 3t$ and $t = 4$, then $x = 3 \cdot 4$.

3. Complete: The median to the base of an isosceles triangle __?__ the vertex angle and is __?__ to the base.

4. Given two parallel lines m and n, how many planes contain m and n?

5. a. Is it possible for two lines to be neither intersecting nor parallel? If so, what are the lines called?
   b. Repeat part (a), replacing *lines* with *planes*.

6. If two parallel lines are cut by a transversal and an interior angle formed has measure 50, find the measure of the other interior angle on the same side of the transversal.

7. In plane P, line $j \perp$ line l and line $k \perp$ line l. Can you conclude anything about lines j and k?

8. Is it possible for a triangle to be equiangular and scalene? Write a theorem or corollary that supports your answer.

9. Find the sum of the measures of the angles of a pentagon.

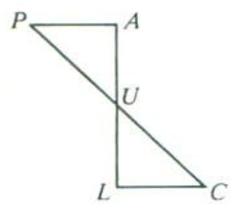
10. Find the measure of each exterior angle of a regular polygon with 12 sides.

11. A certain if-then statement is known to be false. Is the contrapositive of the statement true, false, or sometimes true and sometimes false?

12. Write the converse of the statement "If you are a member of the skiing club, then you enjoy winter weather."

13. It is known that $\triangle ART \cong \triangle DEB$.
   a. $\triangle EBD \cong$ __?__ b. $m\angle R =$ __?__ c. $\overline{DE} \cong$ __?__

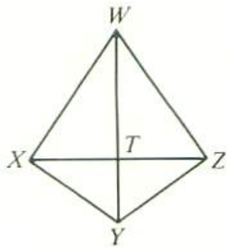
14. Can the given information be used to prove the triangles congruent? If so, which congruence postulate or theorem would you use?



   a. Given: $\overline{PC}$ and $\overline{AL}$ bisect each other.
   b. Given: $\angle P \cong \angle C$; U is the midpoint of $\overline{PC}$.
   c. Given: $\overline{PA} \parallel \overline{LC}$
   d. Given: $\overline{PA} \perp \overline{AL}$; $\overline{LC} \perp \overline{AL}$; $\overline{PU} \cong \overline{UC}$

15. a. If a point lies on the perpendicular bisector of $\overline{AB}$, then the point is equidistant from __?__.
   b. If a point lies on the bisector of $\angle RST$, then the point is equidistant from __?__.

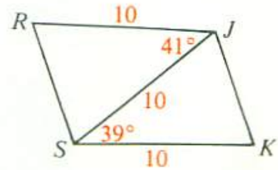
16. Write the reason for each key step.
Given: $\overline{WX} \cong \overline{WZ}$; $\overline{WY} \perp \overline{XZ}$
Prove: $XY = ZY$
a. $\triangle WXT \cong \triangle WZT$
b. $\overline{XT} \cong \overline{TZ}$
c. $\overline{WY}$ is the perpendicular bisector of $\overline{XZ}$.
d. $XY = ZY$



17. Tell whether the statement is *always, sometimes,* or *never* true for a parallelogram $ABCD$ with diagonals that intersect at P.
a. $AB = BC$ **b.** $\overline{AC} \perp \overline{BD}$ **c.** $\angle A$ and $\angle B$ are comp. ∠s.
d. $\angle ADB \cong \angle CBD$ **e.** $\overline{AP} \cong \overline{PC}$ **f.** $\triangle ABC \cong \triangle CDA$

18. In $\triangle RST$, $m\angle R = 64$ and $m\angle S = 54$. Name (a) the longest and (b) the shortest side of $\triangle RST$.

19. **a.** Which segment is longer: $\overline{RS}$ or $\overline{JK}$?
b. Name the theorem that supports your answer.



B 20. Give the best name for quadrilateral $MNOP$.
a. $\overline{MN} \cong \overline{PO}$; $\overline{MN} \parallel \overline{PO}$
b. $\overline{MN} \parallel \overline{PO}$; $\overline{NO} \parallel \overline{MP}$; $\overline{MO} \perp \overline{NP}$
c. $\angle M \cong \angle N \cong \angle O \cong \angle P$
d. $\angle M \cong \angle N$; $\angle O \cong \angle P$; $\angle M \not\cong \angle O$
e. $MNOP$ is a rectangle with $MN = NO$.

21. In $\triangle SUN$, $\angle S \cong \angle N$. Given that $SU = 2x + 7$, $UN = 4x - 1$, and $SN = 3x + 4$, find the value of x.

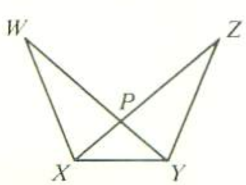
22. The difference between the measures of two supplementary angles is 38. Find the measure of each angle.

23. The lengths of the sides of a triangle are z, $z + 3$, and $z + 6$. What can you conclude about the value of z?

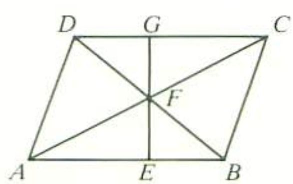
24. M and N are the midpoints of the legs of trapezoid $EFGH$. If bases $\overline{EF}$ and $\overline{HG}$ have lengths $2r + s$ and $4r - 3s$, express the length of $\overline{MN}$ in terms of r and s.

25. B lies between A and C. $AB = 3.2y$, $BC = 2y + 1$, and $AC = 6y - 1$. Is B the midpoint of $\overline{AC}$? Explain.

26. Given: $WP = ZP$; $PY = PX$
Prove: $\angle WXY \cong \angle ZYX$



27. Given: $\overline{AD} \cong \overline{BC}$; $\overline{AD} \parallel \overline{BC}$
Prove: $\overline{EF} \cong \overline{FG}$



An interesting example of similarity is found in this unusual aquarium that is a scale model of a kitchen. Each object placed in the tank was made to the same scale. Wallpaper and window frames, in an appropriate scale, were applied to the outside of the tank.

Similar Polygons

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