**63.** (a) Substituting $x^2$ for $x$ in the Maclaurin series for $\sin x$ given at the end of Section 9.2,

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots + (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

Integrating term-by-term and observing that the constant term is 0,

$$\int_0^x \sin t^2\,dt = \frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} - \cdots + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \cdots$$

(b) $$\int_0^1 \sin x\,dx = \frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \cdots + (-1)^n \frac{1}{(4n+3)(2n+1)!} + \cdots.$$

Since the third term is $\frac{1}{11(5!)} = \frac{1}{1320} < 0.001$, it suffices to use the first two nonzero terms (through degree 7).

(c) NINT($\sin x^2, x, 0, 1$) $\approx 0.31026830$

(d) $\frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \frac{1}{15(7!)} = \frac{258{,}019}{831{,}600} \approx 0.31026816$

This is within $1.5 \times 10^{-7}$ of the answer in (c).

**64.** (a) Let $f(x) = x^2e^x\,dx$.

$$\int_0^1 x^2e^x\,dx = \int_0^1 f(x)\,dx$$
$$\approx \frac{h}{2}\left[f(0) + 2f(0.5) + f(1)\right]$$
$$= \frac{1}{4}\left[0 + 2\frac{e^{0.5}}{4} + e\right]$$
$$= \frac{e^{0.5}}{8} + \frac{e}{4}$$
$$\approx 0.88566$$

(b) $$x^2e^x = x^2\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$
$$= x^2 + x^3 + \frac{x^4}{2!} + \cdots + \frac{x^{n+2}}{n!} + \cdots$$

$$P_4(x) = x^2 + x^3 + \frac{x^4}{2}$$

$$\int_0^1 P_4(x) = \left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{41}{60} \approx 0.68333$$

(c) Since $f$ is concave up, the trapezoids used to estimate the area lie above the curve, and the estimate is too large.

(d) Since all the derivatives are positive (and $x > 0$), the remainder, $R_n(x)$, must be positive. This means that $P_n(x)$ is smaller than $f(x)$.

(e) Let $u = x^2 \quad dv = e^x\,dx$

$du = 2x\,dx \quad v = e^x$

$$\int x^2e^x\,dx = x^2e^x - \int 2x\,e^x\,dx$$

Let $u = 2x \quad dv = e^x\,dx$

$du = 2\,dx \quad v = e^x$

$$x^2e^x - \int 2xe^x\,dx = x^2e^x - \left[2xe^x - \int 2e^x\,dx\right]$$
$$= x^2e^x - 2xe^x + 2e^x + C$$
$$= (x^2 - 2x + 2)e^x + C$$

$$\int_0^1 x^2e^x\,dx = (x^2 - 2x + 2)e^x\Big]_0^1 = e - 2 \approx 0.71828$$

**65.** (a) Because $[\$1000(1.08)^{-n}](1.08)^n = \$1000$ will be available after $n$ years.

(b) Assume that the first payment goes to the charity at the end of the first year.

$1000(1.08)^{-1} + 1000(1.08)^{-2} + 1000(1.08)^{-3} + \cdots$

(c) This is a geometric series with sum equal to $\frac{1000/1.08}{1 - (1/1.08)} = \frac{1000}{0.08} = 12{,}500$. This means that \$12,500 should be invested today in order to completely fund the perpetuity forever.

**66.** We again assume that the first payment occurs at the end of the year.

Present value $= 1000(1.06)^{-1} + 1000(1.06)^{-2} + 1000(1.06)^{-3} + \cdots$

$$= \frac{1000/1.06}{1 - (1/1.06)} = \frac{1000}{1.06 - 1} \approx 16{,}666.67$$

The present value is \$16,666.67.

**67. (a)**

| Sequence of Tosses | Payoff ($) | Probability | Term of Series |
|---|---|---|---|
| T | 0 | $\frac{1}{2}$ | $0\left(\frac{1}{2}\right)$ |
| HT | 1 | $\left(\frac{1}{2}\right)^2$ | $1\left(\frac{1}{2}\right)^2$ |
| HHT | 2 | $\left(\frac{1}{2}\right)^3$ | $2\left(\frac{1}{2}\right)^3$ |
| HHHT | 3 | $\left(\frac{1}{2}\right)^4$ | $3\left(\frac{1}{2}\right)^4$ |
| ⋮ | ⋮ | ⋮ | ⋮ |

Expected payoff

$$= 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 + \cdots$$

**(b)** $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$

**(c)** $\frac{x^2}{(1-x)^2} = x^2(1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots)$

$= x^2 + 2x^3 + 3x^4 + \cdots + nx^{n+1} + \cdots$

**(d)** If $x = \frac{1}{2}$, the formula in part (c) matches the nonzero terms of the series in part (a). Since $\frac{(1/2)^2}{[1-(1/2)]^2} = 1$, the expected payoff is $1.

**68. (a)** The area of an equilateral triangle whose sides have length $s$ is $\frac{1}{2}(s)\left(\frac{\sqrt{3}s}{2}\right) = \frac{s^2\sqrt{3}}{4}$. The sequence of areas removed from the original triangle is

$$\frac{b^2\sqrt{3}}{4} + 3\left(\frac{b}{2}\right)^2\frac{\sqrt{3}}{4} + 9\left(\frac{b}{4}\right)^2\frac{\sqrt{3}}{4} + \cdots$$

$$+ 3^n\left(\frac{b}{2^n}\right)^2\frac{\sqrt{3}}{4} + \cdots \text{ or}$$

$$\frac{b^2\sqrt{3}}{4} + \frac{3b^2\sqrt{3}}{4^2} + \frac{3^2b^2\sqrt{3}}{4^3} + \cdots + \frac{3^nb^2\sqrt{3}}{4^{n+1}} + \cdots.$$

**(b)** This is a geometric series with initial term $a = \frac{b^2\sqrt{3}}{4}$ and common ratio $r = \frac{3}{4}$, so the sum is $\frac{b^2\sqrt{3}/4}{1-(3/4)} = b^2\sqrt{3}$, which is the same as the area of the original triangle.

**(c)** No, not every point is removed. For example, the vertices of the original triangle are not removed. But the remaining points are "isolated" enough that there are no regions and hence no area remaining.

**69.** $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

Differentiate both sides.

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

Substitute $x = \frac{1}{2}$ to get the desired result.

**70. (a)** Note that $\sum_{n=1}^{\infty} x^{n+1}$ is a geometric series with first term $a = x^2$ and common ratio $r = x$, which explains the identity $\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x}$ (for $|x| < 1$).

Differentiate.

$$\sum_{n=1}^{\infty}(n+1)x^n = \frac{(1-x)(2x)-(x^2)(-1)}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2}$$

Differentiate again.

$$\sum_{n=1}^{\infty} n(n+1)x^{n-1}$$

$$= \frac{(1-x)^2(2-2x)-(2x-x^2)(2)(1-x)(-1)}{(1-x)^4}$$

$$= \frac{(1-x)(2-2x)+2(2x-x^2)}{(1-x)^3}$$

$$= \frac{2}{(1-x)^3}$$

Multiply by $x$.

$$\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$$

Replace $x$ by $\frac{1}{x}$.

$$\sum_{n=1}^{\infty}\frac{n(n+1)}{x^n} = \frac{\frac{2}{x}}{\left(1-\frac{1}{x}\right)^3} = \frac{2x^2}{(x-1)^3}, |x| > 1$$

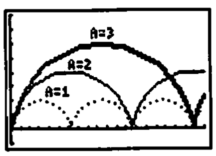
**(b)** Solve $x = \frac{2x^2}{(x-1)^3}$ to get $x \approx 2.769$ for $x > 1$.

# Chapter 10
# Vectors

## ■ Section 10.1 Parametric Functions
(pp. 513–520)

### Exploration 1 Investigating Cycloids

1.




[0, 20] by [−1, 8]

2. $x = 2na\pi$ for any integer $n$.
3. $a > 0$ and $1 - \cos t \geq 0$ so $y \geq 0$.
4. An arch is produced by one complete turn of the wheel. Thus, they are congruent.
5. The maximum value of $y$ is $2a$ and occurs when $x = (2n + 1)a\pi$ for any integer $n$.
6. The function represented by the cycloid is periodic with period $2a\pi$, and each arch represents one period of the graph. In each arch, the graph is concave down, has an absolute maximum of $2a$ at the midpoint, and an absolute minimum of 0 at the two endpoints.

## Quick Review 10.1

1. $(\cos(0), \sin(0)) = (1, 0)$
2. $\left(\cos\left(\frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right)\right) = (0, -1)$
3. $x^2 + y^2 = 1$ (since $\cos^2 t + \sin^2 t = 1$)
4. The portion in the first three quadrants, moving counterclockwise as $t$ increases.
5. $x = t,\ y = t^2 + 1,\ -1 \le t \le 3$
6. The graph is a circle with radius 2 centered at (2, 3). Modify the $x = \cos t$, $y = \sin t$ parameterization correspondingly:
   $x = 2\cos t + 2,\ y = 2\sin t + 3,\ 0 \le t \le 2\pi.$
7. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t}{-2\sin t}$,
   which at $t = \frac{3\pi}{4}$ equals $\frac{3(-\sqrt{2}/2)}{-2(\sqrt{2}/2)} = \frac{3}{2}$.
8. $y = \frac{3}{2}x + C$. For $t = \frac{3\pi}{4}$, $x = -\sqrt{2}$ and $y = \frac{3\sqrt{2}}{2}$, so
   $\frac{3\sqrt{2}}{2} = \frac{3}{2}(-\sqrt{2}) + C$ and $C = 3\sqrt{2}$.
   Thus, $y = \frac{3}{2}x + 3\sqrt{2}$.
9. $y = -\frac{2}{3}x + C$. For $t = \frac{3\pi}{4}$, $x = -\sqrt{2}$ and $y = \frac{3\sqrt{2}}{2}$, so
   $\frac{3\sqrt{2}}{2} = -\frac{2}{3}(-\sqrt{2}) + C$ and $C = \frac{5\sqrt{2}}{6}$.
   Thus, $y = -\frac{2}{3}x + \frac{5\sqrt{2}}{6}$.
10. $y' = \frac{3}{2}\sqrt{x}$, so

$$\text{Length} = \int_0^3 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\,dx = \int_0^3 \sqrt{1 + \frac{9}{4}x}\,dx = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^3 = \frac{31^{3/2} - 8}{27}.$$

## Section 10.1 Exercises

1. (a) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{-2\sin t}{4\cos t} = -\frac{1}{2}\tan t$

   (b) $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\frac{1}{2}\sec^2 t}{4\cos t} = -\frac{1}{8}\sec^3 t$

2. (a) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$

   (b) $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{0}{-\sin t} = 0$

3. (a) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt}$

$$= \frac{3/(2\sqrt{3t})}{-1/(2\sqrt{t+1})} = -\frac{3\sqrt{t+1}}{\sqrt{3t}} = -\sqrt{3 + \frac{3}{t}}$$

   (b) $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{3/(2t^2\sqrt{3 + 3/t})}{-1/(2\sqrt{t+1})} = -\frac{3\sqrt{t+1}}{t^2\sqrt{3 + 3/t}}$

$$= -\frac{\sqrt{3}}{t^{3/2}}$$

4. (a) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{1/t}{-1/t^2} = -t$

   (b) $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-1}{-1/t^2} = t^2$

5. (a) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t - 3}$

   (b) $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$

$$= \frac{[(2t-3)(6t) - (3t^2)(2)]/(2t-3)^2}{2t-3} = \frac{12t^2 - 18t - 6t^2}{(2t-3)^3} = \frac{6t^2 - 18t}{(2t-3)^3}$$

6. (a) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{2t - 1}{2t + 1}$

   (b) $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$

$$= \frac{[(2t+1)(2) - (2t-1)(2)]/(2t+1)^2}{2t+1} = \frac{4}{(2t+1)^3}$$

7. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$

   (a) $-\cot = 0$ when $t = \frac{\pi}{2} + k\pi$ ($k$ any integer). Then
   $(x, y) = \left(2 + \cos\left(\frac{\pi}{2} + k\pi\right), -1 + \sin\left(\frac{\pi}{2} + k\pi\right)\right)$
   $= (2, -1 \pm 1)$. The points are (2, 0) and (2, −2).

   (b) $-\cot$ is undefined when $t = k\pi$ ($k$ any integer). Then
   $(x, y) = (2 + \cos(k\pi), -1 + \sin(k\pi)) = (2 \pm 1, -1)$.
   The points are (1, −1) and (3, −1).

**8.** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \csc t$

**(a)** Nowhere, since $\csc t$ never equals zero.

**(b)** $\csc t$ is undefined when $t = k\pi$ ($k$ any integer). Then $(x, y) = (\sec(k\pi), \tan(k\pi)) = (\pm 1, 0)$. The points are $(1, 0)$ and $(-1, 0)$.

**9.** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 4}{-1} = 4 - 3t^2$

**(a)** $4 - 3t^2 = 0$ when $t = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$.

Then $(x, y) = \left(2 \mp \frac{2}{\sqrt{3}}, \pm\left(\frac{2}{\sqrt{3}}\right)^3 \mp 4\left(\frac{2}{\sqrt{3}}\right)\right)$
$= \left(2 \mp \frac{2}{\sqrt{3}}, \pm\frac{8}{3\sqrt{3}} \mp \frac{8}{\sqrt{3}}\right)$,
which evaluates to $\approx (0.845, -3.079)$ and $\approx (3.155, 3.079)$.

**(b)** Nowhere, since $4 - 3t^2$ is never undefined.

**10.** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t}{-3\sin t} = -\cot t$

**(a)** $-\cot t = 0$ when $t = \frac{\pi}{2} + k\pi$ ($k$ any integer).

Then $(x, y)$
$= \left(-2 + 3\cos\left(\frac{\pi}{2} + k\pi\right), 1 + 3\sin\left(\frac{\pi}{2} + k\pi\right)\right)$
$= (-2, 1 \pm 3)$. The points are $(-2, 4)$ and $(-2, -2)$.

**(b)** $-\cot t$ is undefined when $t = k\pi$ ($k$ any integer). Then

$(x, y) = (-2 + 3\cos(k\pi), 1 + 3\sin(k\pi))$

$= (-2 \pm 3, 1)$. The points are $(1, 1)$ and $(-5, 1)$.

**11.** $x' = -\sin t$, $y' = 1 + \cos t$, so

$$\begin{aligned}\text{length} &= \int_0^\pi \sqrt{(-\sin t)^2 + (1 + \cos t)^2}\,dt \\ &= \int_0^\pi \sqrt{2(1 + \cos t)}\,dt \\ &= \int_0^\pi \sqrt{4\cos^2\left(\frac{t}{2}\right)}\,dt \\ &= \int_0^\pi 2\cos\left(\frac{t}{2}\right)dt \\ &= 2\left[2\sin\left(\frac{t}{2}\right)\right]_0^\pi = 4\end{aligned}$$

**12.** $x' = \sqrt{2t + 3}$, $y' = 1 + t$, so

$$\begin{aligned}\text{Length} &= \int_0^3 \sqrt{(\sqrt{2t + 3})^2 + (1 + t)^2}\,dt \\ &= \int_0^3 (t + 2)\,dt = \left[\frac{1}{2}t^2 + 2t\right]_0^3 = \frac{21}{2}\end{aligned}$$

**13.** $x' = t^2$, $y' = t$, so

$$\begin{aligned}\text{Length} &= \int_0^1 \sqrt{(t^2)^2 + t^2}\,dt \\ &= \int_0^1 t\sqrt{t^2 + 1}\,dt \\ &= \left[\frac{1}{3}(t^2 + 1)^{3/2}\right]_0^1 \\ &= \frac{1}{3}(2^{3/2} - 1) \\ &= \frac{2\sqrt{2} - 1}{3} \approx 0.609\end{aligned}$$

**14.** $x' = 8t\cos t$, $y' = 8t\sin t$, so

$$\begin{aligned}\text{Length} &= \int_0^{\pi/2} \sqrt{(8t\cos t)^2 + (8t\sin t)^2}\,dt \\ &= \int_0^{\pi/2} 8t\,dt \\ &= \left[4t^2\right]_0^{\pi/2} = \pi^2\end{aligned}$$

**15.** $x' = \frac{\sec t\tan t + \sec^2 t}{\sec t + \tan t} - \cos t = \sec t - \cos t$,

$y' = -\sin t$, so

$$\begin{aligned}\text{Length} &= \int_0^{\pi/3} \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}\,dt \\ &= \int_0^{\pi/3} \sqrt{\sec^2 t - 1}\,dt \\ &= \int_0^{\pi/3} \tan t\,dt \\ &= \left[\ln|\sec t|\right]_0^{\pi/3} = \ln 2\end{aligned}$$

**16.** $x' = e^t - 2t$, $y' = 1 - e^{-t}$, so

$$\text{Length} = \int_{-1}^2 \sqrt{(e^t - 2t)^2 + (1 - e^{-t})^2}\,dt,$$

which using NINT evaluates to $\approx 4.497$.

**17.** $x' = -\sin t$, $y' = \cos t$, so

$$\begin{aligned}\text{Area} &= \int_0^{2\pi} 2\pi(2 + \sin t)\sqrt{(-\sin t)^2 + \cos^2 t}\,dt \\ &= 2\pi\int_0^{2\pi} (2 + \sin t)\,dt \\ &= 2\pi\left[2t - \cos t\right]_0^{2\pi} = 8\pi^2\end{aligned}$$

**18.** $x' = \sqrt{t}$, $y' = \frac{1}{\sqrt{t}}$, so

$$\begin{aligned}\text{Area} &= \int_0^2 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{(\sqrt{t})^2 + \left(\frac{1}{\sqrt{t}}\right)^2}\,dt \\ &= \frac{4\pi}{3}\int_0^2 t\sqrt{t^2 + 1}\,dt \\ &= \frac{4\pi}{3}\left[\frac{1}{3}(t^2 + 1)^{3/2}\right]_0^2 \\ &= \frac{4\pi(5\sqrt{5} - 1)}{9} \approx 14.214\end{aligned}$$

**19.** $x' = 1$, $y' = 2t$, so

$$\text{Area} = \int_0^3 2\pi(t + 1)\sqrt{1 + (2t)^2}\,dt,$$

which using NINT evaluates to $\approx 178.561$.

**20.** $x' = \sec t - \cos t$ (see Ex. 15), $y' = -\sin t$, so

$$\begin{aligned}\text{Length} &= \int_0^{\pi/3} 2\pi(\cos t)\sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}\,dx \\ &= \int_0^{\pi/3} 2\pi \cos t \sqrt{\tan^2 t}\,dt \\ &= 2\pi\int_0^{\pi/3} \sin t\,dt \\ &= 2\pi\Big[-\cos t\Big]_0^{\pi/3} = \pi.\end{aligned}$$

**21.** **(a)** $x(t) = 2t$, $y(t) = t + 1$, $0 \le t \le 1$

**(b)** $x' = 2$, $y' = 1$, so

$$\begin{aligned}\text{Area} &= \int_0^1 2\pi(t+1)\sqrt{2^2 + 1^2}\,dt \\ &= 2\pi\sqrt{5}\int_0^1 (t+1)\,dt \\ &= 2\pi\sqrt{5}\left[\frac{1}{2}t^2 + t\right]_0^1 \\ &= 3\pi\sqrt{5}\end{aligned}$$

**(c)** Slant height $= \sqrt{2^2 + 1^2} = \sqrt{5}$, so

Area $= \pi(1 + 2)\sqrt{5} = 3\pi\sqrt{5}$

**22.** **(a)** Because these values for $x(t)$ and $y(t)$ satisfy $y = \frac{r}{h}x$, which is the equation of the line through the origin and $(h, r)$, and this range of $t$-values gives the correct initial and terminal points.

**(b)** $x' = h$, $y' = r$, so

$$\begin{aligned}\text{Area} &= \int_0^1 2\pi(rt)\sqrt{h^2 + r^2}\,dt \\ &= \pi r\sqrt{h^2 + r^2}\Big[t^2\Big]_0^1 \\ &= \pi r\sqrt{r^2 + h^2}.\end{aligned}$$

**(c)** Slant height $= \sqrt{r^2 + h^2}$, so Area $= \pi r\sqrt{r^2 + h^2}$

**23.** **(a)** $x' = -2\sin 2t$, $y' = 2\cos 2t$, so

$$\begin{aligned}\text{Length} &= \int_0^{\pi/2} (\sqrt{-2\sin 2t)^2 + (2\cos 2t)^2}\,dt \\ &= \int_0^{\pi/2} 2\,dt = \pi.\end{aligned}$$

**(b)** $x' = \pi\cos \pi t$, $y' = -\pi\sin \pi t$, so

$$\begin{aligned}\text{Length} &= \int_{-1/2}^{1/2} \sqrt{(\pi\cos \pi t)^2 + (-\pi\sin \pi t)^2}\,dt \\ &= \int_{-1/2}^{1/2} \pi\,dt = \pi.\end{aligned}$$

**24.** $x' = -3\sin t$, $y' = 4\cos t$, so

$$\text{Length} = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (4\cos t)^2}\,dt$$

which using NINT evaluates to $\approx 22.103$.

**25.** In the first integral, replace $t$ with $x$. Then $\frac{dx}{dt}$ becomes $\frac{dx}{dx} = 1$.

**26.** Parameterize the curve as $x = g(y)$, $y = y$, $c \le y \le d$. The parameter is $y$ itself, so replace $t$ with $y$ in the general formula. Then $\frac{dy}{dt}$ becomes $\frac{dy}{dy} = 1$.

**27.** $x' = t$, $y' = \sqrt{2t + 1}$, so

$$\begin{aligned}\text{Total length} &= \int_0^4 \sqrt{t^2 + (\sqrt{2t+1})^2}\,dt \\ &= \int_0^4 (t+1)\,dt \\ &= \left[\frac{1}{2}t^2 + t\right]_0^4 = 12.\end{aligned}$$

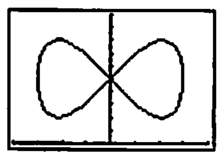
Now solve $\left[\frac{1}{2}t^2 + t\right]_0^m = \frac{12}{2}$ for $m$:

$\frac{1}{2}m^2 + m = 6$, or $m^2 + 2m - 12 = 0$, and

$m = \frac{-2 \pm \sqrt{4 + 48}}{2} = -1 \pm \sqrt{13}$. Take the positive solution. The midpoint is at $t = \sqrt{13} - 1$, which gives

$(x, y) = \left(\frac{(\sqrt{13} - 1)^2}{2}, \frac{1}{3}(2\sqrt{13} - 1)^{3/2}\right) \approx (3.394, 5.160)$.

**28.**



[−4, 4] by [0, 10]

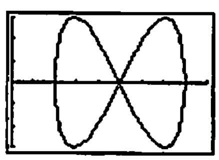
Use the right half of the curve, $0 \le t \le \pi$.

$x' = 3\cos t$, $y' = 6\cos 2t$, so

$$\text{Area} = \int_0^{\pi} 2\pi(3\sin t)\sqrt{(3\cos t)^2 + (6\cos 2t)^2}\,dt,$$

which using NINT evaluates to $\approx 159.485$.

**29.**



[0, 9] by [−3, 3]

Use the top half of the curve, and make use of the shape's symmetry.

$x' = 3\cos t$, $y' = 6\cos 2t$, so

$$\text{Area} = 2\int_0^{\pi/2} 2\pi(3\sin 2t)\sqrt{(3\cos t)^2 + (6\cos 2t)^2}\,dt$$

which using NINT, evaluates to $\approx 144.513$.

30. $y = 0$ for $t = 0$ and $t = 2\pi$. $x' = a - a\cos t$, $y' = a\sin t$,

so

$$\text{Area} = \int_0^{2\pi} 2\pi[a(1-\cos t)]\sqrt{(a - a\cos t)^2 + (a\sin t)^2}\,dt$$
$$= 2\pi a^2 \int_0^{2\pi} (1-\cos t)\sqrt{2 - 2\cos t}\,dt$$
$$= 2\pi a^2 \int_0^{2\pi} \left(2\sin^2\left(\frac{t}{2}\right)\right)\sqrt{4\sin^2\left(\frac{t}{2}\right)}\,dt$$
$$= 8\pi a^2 \int_0^{2\pi} \sin^3\left(\frac{t}{2}\right)dt$$
$$= 8\pi a^2 \int_0^{2\pi} \left(1 - \cos^2\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right)dt$$
$$= 8\pi a^2 \left[-2\cos\left(\frac{t}{2}\right) + \frac{2}{3}\cos^3\left(\frac{t}{2}\right)\right]_0^{2\pi}$$
$$= \frac{64\pi a^2}{3}$$

31. $\frac{dx}{dt} = a(1 - \cos t)$

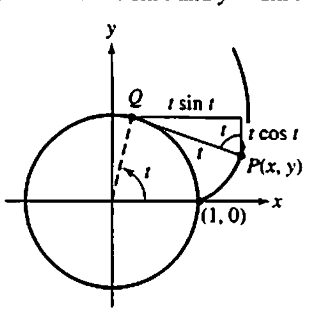
(Note: integrate with respect to $x$ from 0 to $2a\pi$; integrate with respect to $t$ from 0 to $2\pi$.)

$$\text{Area} = \int_0^{2a\pi} y\,dx$$
$$= \int_0^{2a\pi} a(1-\cos t)a(1-\cos t)\,dt$$
$$= a^2\int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt$$
$$= a^2\left[t - 2\sin t + \frac{t}{2} + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi a^2$$

32. $\frac{dx}{dt} = a(1 - \cos t)$, so

$$\text{Volume} = \int_0^{2\pi} \pi[a(1-\cos t)]^2 a(1-\cos t)\,dt$$
$$= \pi a^3\int_0^{2\pi}(1 - 3\cos t + 3\cos^2 t - \cos^3 t)\,dt$$
$$= \pi a^3\left[t - 3\sin t + \frac{3}{2}t + \frac{3}{4}\sin 2t - \left(\sin t - \frac{1}{3}\sin^3 t\right)\right]_0^{2\pi}$$
$$= 5\pi^2 a^3$$

33. (a) $\overline{QP}$ has length $t$, so $P$ can be obtained by starting at $Q$ and moving $t\sin t$ units right and $t\cos t$ units downward.
(If either quantity is negative, the corresponding direction is reversed.) Since $Q = (\cos t, \sin t)$, the coordinates of $P$ are
$x = \cos t + t\sin t$ and $y = \sin t - t\cos t$.




(b) $x' = t\cos t$, $y' = t\sin t$, so

$$\text{Length} = \int_0^{2\pi}\sqrt{(t\cos t)^2 + (t\sin t)^2}\,dt = \int_0^{2\pi} t\,dt$$
$$= 2\pi^2$$

34. All distances are $a$ times as big as before.

(a) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

(b) Length $= 2a\pi^2$

For exercises 35–38, $x' = v_0\cos\theta$ and $y' = v_0\sin\theta - 32t$, and $y = 0$ for $t = 0$ or $t = \frac{v_0\sin\theta}{16}$. The maximum height is attained in mid-flight at $t = \frac{v_0\sin\theta}{32}$. To find the path length, evaluate $\int_0^{v_0\sin\theta/16}\sqrt{(v_0\cos\theta)^2 + (v_0\sin\theta - 32t)^2}\,dt$ using NINT. To find the

maximum height, calculate

$$y_{max} = (v_0\sin\theta)\left(\frac{v_0\sin\theta}{32}\right) - 16\left(\frac{v_0\sin\theta}{32}\right)^2.$$

35. (a) The projectile hits the ground when $y = 0$.

$$y = t(150\sin 20° - 16t) = 0$$
$$t = 0 \text{ or } t = \frac{75}{8}\sin 20° \approx 3.206$$
$$x' = 150\cos 20°,\ y' = 150\sin 20° - 32t$$
$$\text{Length} = \int_0^{(75\sin 20°)/8}\sqrt{(150\cos 20°)^2 + (150\sin 20° - 32t)^2}$$

$dt$ which, using NINT, evaluates to $\approx 461.749$ ft

(b) The maximum height of the projectile occurs when

$y' = 0$,

so $t = \frac{75}{16}\sin 20°$, $y\left(\frac{75}{16}\sin 20°\right) \approx 41.125$ ft

36. (a) $\approx 641.236$ ft

(b) $\frac{5625}{64} \approx 87.891$ ft

37. (a) $\approx 840.421$ ft

(b) $\frac{16{,}875}{64} \approx 263.672$ ft

38. (a) It is not necessary to use NINT.

$$\text{Length} = \int_0^{75/8}(150 - 32t)\,dt = \left[150t - 16t^2\right]_0^{75/8}$$
$$= \frac{5625}{8} = 703.125 \text{ ft}$$

(b) $\frac{5625}{16} = 351.5625$

39. In the integral $\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, replace $t$ with $x$ and $y$ with $f(x)$.

Then $\frac{dx}{dt}$ becomes $\frac{dx}{dx} = 1$.

**40.** $\frac{dy}{dx} = e^x$, so Area $= \int_0^3 2\pi e^x\sqrt{1 + (e^x)^2}\,dx$ which using NINT evaluates to $\approx 1273.371$.

**41.** $\frac{dy}{dx} = -\frac{1}{x^2}$, so Area $= \int_1^4 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx$, which, using NINT, evaluates to $\approx 9.417$.

**42.** $\frac{dy}{dx} = (\ln 2)2^x - (\ln 2)2^{-x} = (\ln 2)(2^x - 2^{-x})$, so
Area $= \int_{-2}^{2} 2\pi(2^x + 2^{-x})\sqrt{1 + (\ln 2)^2(2^x - 2^{-x})^2}\,dx$ which using NINT evaluates to $\approx 116.687$.

## ■ Section 10.2 Vectors in the Plane
(pp. 520–529)

### Quick Review 10.2

**1.** $\sqrt{(5-1)^2 + (3-2)^2} = \sqrt{17}$

**2.** $\frac{3-2}{5-1} = \frac{1}{4}$

**3.** Solve $\frac{3-b}{5-3} = -4$: $b = 11$.

**4.** Slope of $\overline{AB}$ = Slope of $\overline{CD}$, so $\frac{3-0}{1-0} = \frac{3-0}{5-a}$ and $a = 4$.

**5.** Slope of $\overline{AB}$ = Slope of $\overline{CD}$, so $\frac{5-1}{3-1} = \frac{2-b}{6-8}$ = and $b = 6$.

**6.** **(a)** $\theta = 120°$
**(b)** $\theta = \frac{2\pi}{3}$

**7.** **(a)** $\theta = -30°$
**(b)** $\theta = -\frac{\pi}{6}$

**8.** **(a)** $\theta = -45°$
**(b)** $\theta = -\frac{\pi}{4}$

**9.** $c^2 = 3^2 + 5^2 - 2(3)(5)\cos(30°) = 34 - 15\sqrt{3}$, so
$c = \sqrt{34 - 15\sqrt{3}} \approx 2.832$

**10.** $24^2 = 27^2 + 19^2 - 2(27)(19)\cos\theta$, so
$\cos\theta = \frac{24^2 - 27^2 - 19^2}{-2(27)(19)} = \frac{257}{513}$ and
$\theta = \cos^{-1}\frac{257}{513} \approx 1.046$ radians or $59.935°$.

### Section 10.2 Exercises

**1.** **(a)** $\langle 3(3), 3(-2)\rangle = \langle 9, -6\rangle$
**(b)** $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

**2.** **(a)** $\langle -2(-2), -2(5)\rangle = \langle 4, -10\rangle$
**(b)** $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

**3.** **(a)** $\langle 3 + (-2), -2 + 5\rangle = \langle 1, 3\rangle$
**(b)** $\sqrt{1^2 + 3^2} = \sqrt{10}$

**4.** **(a)** $\langle 3 - (-2), -2 - 5\rangle = \langle 5, -7\rangle$
**(b)** $\sqrt{5^2 + (-7)^2} = \sqrt{74}$

**5.** **(a)** $2\mathbf{u} = \langle 2(3), 2(-2)\rangle = \langle 6, -4\rangle$
$3\mathbf{v} = \langle 3(-2), 3(5)\rangle = \langle -6, 15\rangle$
$2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-6), -4 - 15\rangle = \langle 12, -19\rangle$
**(b)** $\sqrt{12^2 + (-19)^2} = \sqrt{505}$

**6.** **(a)** $-2\mathbf{u} = \langle -2(3), -2(-2)\rangle = \langle -6, 4\rangle$
$5\mathbf{v} = \langle 5(-2), 5(5)\rangle = \langle -10, 25\rangle$
$-2\mathbf{u} + 5\mathbf{v} = \langle -6 + (-10), 4 + 25\rangle = \langle -16, 29\rangle$
**(b)** $\sqrt{(-16)^2 + 29^2} = \sqrt{1097}$

**7.** **(a)** $\frac{3}{5}\mathbf{u} = \left\langle \frac{3}{5}(3), \frac{3}{5}(-2)\right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5}\right\rangle$
$\frac{4}{5}\mathbf{v} = \left\langle \frac{4}{5}(-2), \frac{4}{5}(5)\right\rangle = \left\langle -\frac{8}{5}, 4\right\rangle$
$\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v} = \left\langle \frac{9}{5} + \left(-\frac{8}{5}\right), -\frac{6}{5} + 4\right\rangle = \left\langle \frac{1}{5}, \frac{14}{5}\right\rangle$
**(b)** $\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \frac{\sqrt{197}}{5}$

**8.** **(a)** $-\frac{5}{13}\mathbf{u} = \left\langle -\frac{5}{13}(3), -\frac{5}{13}(-2)\right\rangle = \left\langle -\frac{15}{13}, \frac{10}{13}\right\rangle$
$\frac{12}{13}\mathbf{v} = \left\langle \frac{12}{13}(-2), \frac{12}{13}(5)\right\rangle = \left\langle -\frac{24}{13}, \frac{60}{13}\right\rangle$
$-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = \left\langle -\frac{15}{13} + \left(-\frac{24}{13}\right), \frac{10}{13} + \frac{60}{13}\right\rangle = \left\langle -3, \frac{70}{13}\right\rangle$
**(b)** $\sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \frac{\sqrt{6421}}{13}$

**9.** $\langle 2 - 1, -1 - 3\rangle = \langle 1, -4\rangle$

**10.** $\left\langle \frac{2 + (-4)}{2} - 0, \frac{-1 + 3}{2} - 0\right\rangle = \langle -1, 1\rangle$

**11.** $\langle 0 - 2, 0 - 3\rangle = \langle -2, -3\rangle$

**12.** $\overrightarrow{AB} = \langle 2 - 1, 0 - (-1)\rangle = \langle 1, 1\rangle$
$\overrightarrow{CD} = \langle -2 - (-1), 2 - 3\rangle = \langle -1, -1\rangle$
$\overrightarrow{AB} + \overrightarrow{CD} = \langle 1 + (-1), 1 + (-1)\rangle = \langle 0, 0\rangle$

**13.** $\left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$

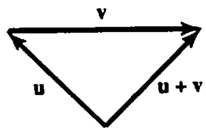
**14.** $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$

**15.** This is the unit vector which makes an angle of $120 + 90 = 210°$ with the positive $x$-axis;
$\langle \cos 210°, \sin 210°\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$

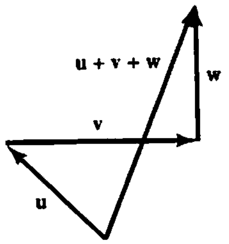
**16.** $\langle \cos 135°, \sin 135°\rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

**17.** The vector **v** is horizontal and 1 in. long. The vectors **u** and **w** are $\frac{11}{16}$ in. long. **w** is vertical and **u** makes a 45° angle with the horizontal. All vectors must be drawn to scale.

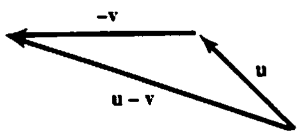
(a)



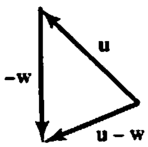

(b)




(c)

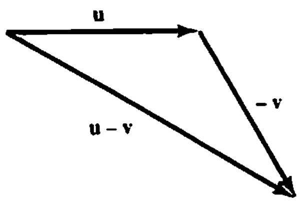



(d)

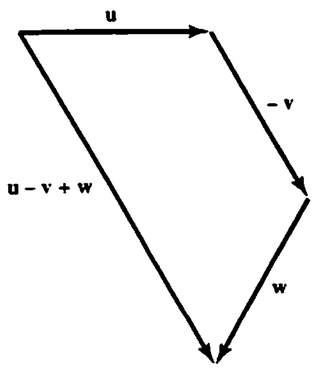



**18.** The angles between the vectors is 120° and vector **u** is horizontal. They are all 1 in. long. Draw to scale.

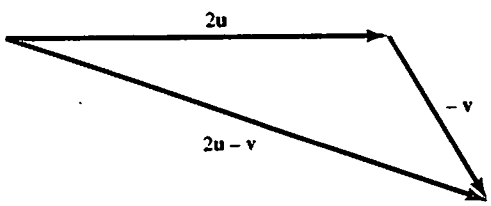
(a)



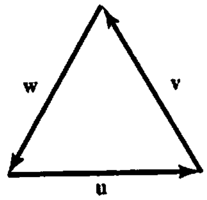

(b)




(c)




(d)




$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$

**19.** $\sqrt{3^2 + 4^2} = 5; \frac{1}{5}\langle 3, 4\rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

**20.** $\sqrt{4^2 + (-3)^2} = 5; \frac{1}{5}\langle 4, -3\rangle = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$

**21.** $\sqrt{(-15)^2 + 8^2} = 17; \frac{1}{17}\langle -15, 8\rangle = \left\langle -\frac{15}{17}, \frac{8}{17} \right\rangle$

**22.** $\sqrt{(-5)^2 + (-2)^2} = \sqrt{29};$

$\frac{1}{\sqrt{29}}\langle -5, -2\rangle = \left\langle -\frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}} \right\rangle$

**23.** $x' = \frac{1}{2\sqrt{t}}, y' = 1 + \frac{1}{\sqrt{t}}$; for $t = 1, x' = \frac{1}{2}, y' = 2$, and $\sqrt{(x')^2 + (y')^2} = \frac{\sqrt{17}}{2}$.

Tangent: $\pm\frac{2}{\sqrt{17}}\left\langle \frac{1}{2}, 2 \right\rangle = \pm\left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle$.

Normal: $\pm\frac{2}{\sqrt{17}}\left\langle 2, -\frac{1}{2} \right\rangle = \pm\left\langle \frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle$.

**24.** $x' = \frac{1}{t-1}, y' = 1$; for $t = 3, x' = \frac{1}{2}, y' = 1$, and $\sqrt{(x')^2 + (y')^2} = \frac{\sqrt{5}}{2}$.

Tangent: $\pm\frac{2}{\sqrt{5}}\left\langle \frac{1}{2}, 1 \right\rangle = \pm\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.

Normal $= \pm\frac{2}{\sqrt{5}}\left\langle 1, -\frac{1}{2} \right\rangle = \pm\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$.

**25.** $x' = -4 \sin t, y' = 5 \cos t$; for $t = \frac{\pi}{3}, x' = -2\sqrt{3}$, $y' = \frac{5}{2}$, and $\sqrt{(x')^2 + (y')^2} = \frac{\sqrt{73}}{2}$.

Tangent $= \pm\frac{2}{\sqrt{73}}\left\langle -2\sqrt{3}, \frac{5}{2} \right\rangle = \pm\left\langle -\frac{12}{\sqrt{219}}, \frac{5}{\sqrt{73}} \right\rangle$

$\approx \pm\langle -0.811, 0.585\rangle$,

Normal $= \pm\frac{2}{\sqrt{73}}\left\langle \frac{5}{2}, 2\sqrt{3} \right\rangle = \pm\left\langle \frac{5}{\sqrt{73}}, \frac{12}{\sqrt{219}} \right\rangle$

$\approx \pm\langle 0.585, 0.811\rangle$.

**26.** $x' = -3\sin t,\ y' = 3\cos t$; for $t = -\frac{\pi}{4},\ x' = \frac{3}{\sqrt{2}},$

$y' = \frac{3}{\sqrt{2}}$, and $\sqrt{(x')^2 + (y')^2} = 3$.

Tangent: $\pm\frac{1}{3}\left\langle\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right\rangle = \pm\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$.

Normal: $\pm\frac{1}{3}\left\langle\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right\rangle = \pm\left\langle\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$.

**27.** $\overrightarrow{AB} = \langle 3, 1\rangle$, $\overrightarrow{BC} = \langle -1, -3\rangle$, and $\overrightarrow{AC} = \langle 2, -2\rangle$.

$\overrightarrow{BA} = \langle -3, -1\rangle$, $\overrightarrow{CB} = \langle 1, 3\rangle$, and $\overrightarrow{CA} = \langle -2, 2\rangle$.

$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{10}$, $|\overrightarrow{BC}| = |\overrightarrow{CB}| = \sqrt{10}$, and

$|\overrightarrow{AC}| = |\overrightarrow{CA}| = 2\sqrt{2}$.

$$\text{Angle at } A = \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}\right) = \cos^{-1}\left(\frac{3(2) + 1(-2)}{(\sqrt{10})(2\sqrt{2})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°,$$

$$\text{Angle at } B = \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|}\right) = \cos^{-1}\left(\frac{(-1)(-3) + (-3)(-1)}{(\sqrt{10})(\sqrt{10})}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°, \text{ and}$$

$$\text{Angle at } C = \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{|\overrightarrow{CB}||\overrightarrow{CA}|}\right) = \cos^{-1}\left(\frac{1(-2) + 3(2)}{(\sqrt{10})(2\sqrt{2})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°.$$

**28.** $\overrightarrow{AC} = \langle 2, 4\rangle$ and $\overrightarrow{BD} = \langle 4, -2\rangle$.
$\overrightarrow{AC}\cdot\overrightarrow{BD} = 2(4) + 4(-2) = 0$, so the angle measures 90°.

**29. (a)**
$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= u_1(v_1 + w_1) + u_2(v_2 + w_2)\\ &= (u_1v_1 + u_1w_1) + (u_2v_2 + u_2w_2)\\ &= (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2)\\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}\end{aligned}$$

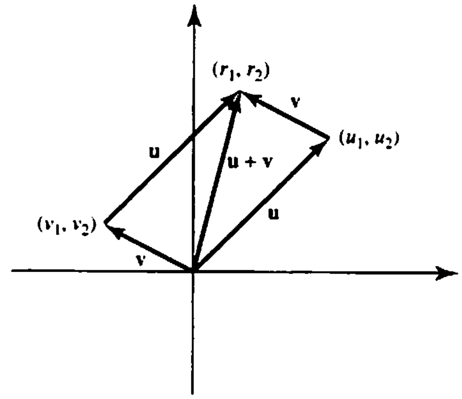
**(b)**
$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2\\ &= (u_1w_1 + v_1w_1) + (u_2w_2 + v_2w_2)\\ &= (u_1w_1 + u_2w_2) + (v_1w_1 + v_2w_2)\\ &= \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}\end{aligned}$$

**30.** $\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 = (\sqrt{u_1^2 + u_2^2})^2 = |\mathbf{u}|^2$

**31.** $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$
$$\begin{aligned}&= (u_1 + v_1)(u_1 - v_1) + (u_2 + v_2)(u_2 - v_2)\\ &= u_1^2 - v_1^2 + u_2^2 - v_2^2\\ &= (u_1^2 + u_2^2) - (v_1^2 + v_2^2)\\ &= |\mathbf{u}|^2 - |\mathbf{v}|^2\end{aligned}$$

**32.** Since **u** and **v** are nonzero, we know that $|\mathbf{u}| \neq 0$ and $|\mathbf{v}| \neq 0$. Therefore, the dot product $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$ is 0 if and only if $\cos\theta = 0$, which occurs if and only if **u** and **v** are orthogonal $\left(\theta = \frac{\pi}{2}\right)$.

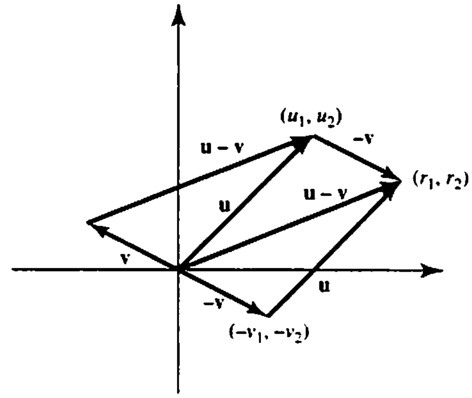
**33.**




$r_1 - v_1 = u_1$ so $r_1 = u_1 + v_1$
$r_2 - v_2 = u_2$ so $r_2 = u_2 + v_2$

**34. (a)** To find $\mathbf{u} - \mathbf{v}$, place both vectors with their initial points at the origin. The vector drawn from the terminal point of **v** to the terminal point of **u** is $\mathbf{u} - \mathbf{v}$. Or, add **u** and $-\mathbf{v}$ according to the parallelogram law.

**(b)**




$r_1 - (-v_1) = u_1 \rightarrow r_1 = u_1 - v_1$
$r_2 - (-v_2) = u_2 \rightarrow r_2 = u_2 - v_2$

**35. (a)** Let $P = (a, b)$ and $Q = (c, d)$. Then
$$\left(\frac{1}{2}\right)\overrightarrow{OP} + \left(\frac{1}{2}\right)\overrightarrow{OQ} = \left(\frac{1}{2}\right)\langle a, b\rangle + \left(\frac{1}{2}\right)\langle c, d\rangle = \left\langle\frac{(a+c)}{2}, \frac{(b+d)}{2}\right\rangle = \overrightarrow{OM}$$

**(b)** $\overrightarrow{OM} = \left(\frac{2}{3}\right)\overrightarrow{OP} + \left(\frac{1}{3}\right)\overrightarrow{OQ}$

**(c)** $\overrightarrow{OM} = \left(\frac{1}{3}\right)\overrightarrow{OP} + \left(\frac{2}{3}\right)\overrightarrow{OQ}$

**35. continued**

**(d)** $M$ is a fraction of the way from $P$ to $Q$. Let $d$ be this fraction. Then

$\overrightarrow{OM} = d\overrightarrow{OQ} + (1 - d)\overrightarrow{OP}$.

Proof: $\overrightarrow{PM} = d\overrightarrow{PQ}$ and $\overrightarrow{MQ} = (1 - d)\overrightarrow{PQ}$,

so $\overrightarrow{PQ} = \frac{1}{d}\overrightarrow{PM}$ and $\overrightarrow{PQ} = \frac{1}{1-d}\overrightarrow{MQ}$.

Therefore, $\frac{1}{d}\overrightarrow{PM} = \frac{1}{1-d}\overrightarrow{MQ}$.

But $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$ and $\overrightarrow{MQ} = \overrightarrow{OQ} - \overrightarrow{OM}$, so

$\frac{1}{d}\overrightarrow{OM} - \frac{1}{d}\overrightarrow{OP} = \frac{1}{1-d}\overrightarrow{OQ} - \frac{1}{1-d}\overrightarrow{OM}$.

Therefore,

$\frac{1}{d}\overrightarrow{OM} + \frac{1}{1-d}\overrightarrow{OM} = \frac{1}{d}\overrightarrow{OP} + \frac{1}{1-d}\overrightarrow{OQ}$.

$\Rightarrow \overrightarrow{OM}\left(\frac{1}{d(1-d)}\right) = \frac{1}{d}\overrightarrow{OP} + \frac{1}{1-d}\overrightarrow{OQ}$

$\Rightarrow \overrightarrow{OM} = (1 - d)\overrightarrow{OP} + d\overrightarrow{OQ}$.

**36.** $\overrightarrow{CA} = -\mathbf{u} - \mathbf{v}$ and $\overrightarrow{CB} = \mathbf{u} - \mathbf{v}$. Since $|\mathbf{v}| = |\mathbf{u}|$, these vectors are orthogonal, as $(-\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$.

**37.** Two adjacent sides of the rhombus can be given by two vectors of the same length, **u** and **v**.
Then the diagonals of the rhombus are $(\mathbf{u} + \mathbf{v})$ and $(\mathbf{u} - \mathbf{v})$.
These two vectors are orthogonal since $|\mathbf{u}| = |\mathbf{v}|$ so
$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$.

**38.** Two adjacent sides of a rectangle can be given by two vectors **u** and **v**. The diagonals are then $(\mathbf{u} + \mathbf{v})$ and $(\mathbf{u} - \mathbf{v})$. These two vectors will be orthogonal if and only if **u** and **v** are the same length, since $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$.

**39.** Let two adjacent sides of the parallelogram be given by two vectors **u** and **v**. The diagonals are then $(\mathbf{u} + \mathbf{v})$ and $(\mathbf{u} - \mathbf{v})$. So the lengths of the diagonals satisfy

$$|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$$

and

$$|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 - 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2.$$

The two lengths will be the same if and only if $\mathbf{u} \cdot \mathbf{v} = 0$, which means that **u** and **v** are perpendicular and the parallelogram is a rectangle.

**40.** The indicated diagonal is $(\mathbf{u} + \mathbf{v})$. The cosine of the angle between the diagonal and **u** is

$$\frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u}}{|\mathbf{u} + \mathbf{v}||\mathbf{u}|} = \frac{|\mathbf{u}|^2 + \mathbf{v} \cdot \mathbf{u}}{|\mathbf{u} + \mathbf{v}||\mathbf{u}|}.$$

But the cosine of the angle between the diagonal and **v** is

$$\frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}}{|\mathbf{u} + \mathbf{v}||\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2}{|\mathbf{u} + \mathbf{v}||\mathbf{v}|}.$$

If **u** and **v** are the same length then these two quantities are equal, and the diagonal makes the same angle with both sides.

**41.** The slopes are the same.

**42.** $\mathbf{v} = \mathbf{0}$ since any other vector has positive magnitude.

**43.** 25° west of north is 90 + 25 = 115° north of east.
$800\langle\cos 115°, \sin 115°\rangle \approx \langle -338.095, 725.046\rangle$

**44.** 10° east of south is 270 + 10 = 280° "north" of east.
$600\langle\cos 280°, \sin 280°\rangle \approx \langle 104.189, -590.885\rangle$

**45.** Initial velocity is 70° north of east:

$325\langle\cos 70°, \sin 70°\rangle \approx \langle 111.157, 305.400\rangle$.

Wind velocity is 130° north of east:

$40\langle\cos 130°, \sin 130°\rangle \approx \langle -25.712, 30.642\rangle$.

Add the two vectors to get $\approx \langle 85.445, 336.042\rangle$.

The speed is the magnitude, $\approx$ 346.735 mph.

The direction is $\tan^{-1}\left(\frac{336.042}{85.445}\right) \approx 75.734°$ north of east, or

$\approx 14.266°$ east of north.

**46.** $|\mathbf{w}|\cos(33° - 15°) = 2.5$ lb, so $|\mathbf{w}| = \frac{2.5 \text{ lb}}{\cos 18°}$.
Then $\mathbf{w} \approx \frac{2.5 \text{ lb}}{\cos 18°}\langle\cos 33°, \sin 33°\rangle \approx \langle 2.205, 1.432\rangle$.

**47.** Juana's pull $= 23\langle\cos 18°, \sin 18°\rangle \approx \langle 21.874, 7.107\rangle$;
Diego's pull $= 18\langle\cos(-15°), \sin(-15°)\rangle$
$\approx \langle 17.387, -4.659\rangle$. Add to get the combined pull of the children: $\approx \langle 39.261, 2.449\rangle$. The puppy pulls with an opposite force of the same magnitude:
$\sqrt{39.261^2 + 2.449^2} \approx 39.337$ lb.

**48. (a)** $7\langle\cos 45°, \sin 45°\rangle$ has its terminal point at $\approx(4.950, 4.950)$.

**(b)** $7\langle\cos 45°, \sin 45°\rangle + 8\langle\cos 210°, \sin 210°\rangle$ has its terminal point at $\approx(-1.978, 0.950)$.

**49.** $\overrightarrow{AB} = \langle -3 - 0, 4 - 0\rangle = \langle -3, 4\rangle = \langle 1 - 4, 5 - 1\rangle = \overrightarrow{CD}$

**50.** $\overrightarrow{AB} = \langle -2 - (-4), -2 - 3\rangle$
$= \langle 2, -5\rangle$
$= \langle 3 - 1, -4 - 1\rangle = \overrightarrow{CD}$

51. $\mathbf{u} = \langle u_1, u_2\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$, $\mathbf{w} = \langle w_1, w_2\rangle$

(i) $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$
$= \langle v_1 + u_1, v_2 + u_2\rangle = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$
$= \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle$
$= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle$
$= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle$
$= \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \langle u_1, u_2\rangle + \langle 0, 0\rangle = \langle u_1 + 0, u_2 + 0\rangle$
$= \langle u_1, u_2\rangle = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = \langle u_1, u_2\rangle + \langle -u_1, -u_2\rangle$
$= \langle u_1 - u_1, u_2 - u_2\rangle = \langle 0, 0\rangle = \mathbf{0}$

(v) $0\mathbf{u} = 0\langle u_1, u_2\rangle = \langle 0u_1, 0u_2\rangle = \langle 0, 0\rangle = \mathbf{0}$

(vi) $1\mathbf{u} = 1\langle u_1, u_2\rangle = \langle 1u_1, 1u_2\rangle = \langle u_1, u_2\rangle = \mathbf{u}$

(vii) $a(b\mathbf{u}) = a(b\langle u_1, u_2\rangle) = a\langle bu_1, bu_2\rangle$
$= \langle abu_1, abu_2\rangle = ab\langle u_1, u_2\rangle = (ab)\mathbf{u}$

(viii) $a(\mathbf{u} + \mathbf{v}) = a\langle u_1 + v_1, u_2 + v_2\rangle$
$= \langle au_1 + av_1, au_2 + av_2\rangle$
$= \langle au_1, au_2\rangle + \langle av_1, av_2\rangle$
$= a\langle u_1, u_2\rangle + a\langle v_1, v_2\rangle$
$= a\mathbf{u} + a\mathbf{v}$

(ix) $(a + b)\mathbf{u} = (a + b)\langle u_1, u_2\rangle$
$= \langle (a + b)u_1, (a + b)u_2\rangle$
$= \langle au_1 + bu_1, au_2 + bu_2\rangle$
$= \langle au_1, au_2\rangle + \langle bu_1, bu_2\rangle = a\mathbf{u} + b\mathbf{u}$

52. Write the two vectors as $a\langle 1, 1\rangle$ and $b\langle 1, -1\rangle$. Then their sum is $\langle a + b, a - b\rangle$, so solve $a + b = 3$, $a - b = 4$ to get $a = \frac{7}{2}$, $b = -\frac{1}{2}$. So $\langle 3, 4\rangle = \left\langle \frac{7}{2}, \frac{7}{2}\right\rangle + \left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle$.

53. (a) Slope $= -\frac{1}{1} = -1$, so $y - y_1 = m(x - x_1)$ becomes $y - 1 = -(x + 2)$ or $y = -x - 1$.

(b) Slope $= 1$, so $y - y_1 = m(x - x_1)$ becomes $y - 1 = x + 2$ or $y = x + 3$.

54. The slopes of the lines are $\frac{3}{4}$ and 1, which means that vectors $\langle 4, 3\rangle$ and $\langle 1, 1\rangle$ are parallel to the respective lines.

$$\theta = \cos^{-1}\left(\frac{4 \cdot 1 + 3 \cdot 1}{5\sqrt{2}}\right) = \cos^{-1}\left(\frac{7\sqrt{2}}{10}\right) \approx 8.130°.$$

## Section 10.3 Vector-valued Functions (pp. 529–539)

### Quick Review 10.3

1. $f'(x) = -\frac{x}{\sqrt{4 - x^2}}$, so for $x = 1$, $f(x) = \sqrt{3}$ and $f'(x) = -\frac{1}{\sqrt{3}}$. Then $y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x - 1)$ or $y = \left(-\frac{1}{\sqrt{3}}\right)x + \frac{4}{\sqrt{3}}$.

2. $f'(x) = -\frac{x}{\sqrt{4 - x^2}}$, so for $x = 1$, $f(x) = \sqrt{3}$ and $f'(x) = -\frac{1}{\sqrt{3}}$. Then $y - \sqrt{3} = \sqrt{3}(x - 1)$ or $y = \sqrt{3}x$.

3. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-4\sin t}$, which for $t = \frac{\pi}{2}$ equals zero.

4. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-4\sin t}$, which for $t = \pi$ is undefined: the tangent line is vertical.

5. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-4\sin t}$, which for $t = \frac{\pi}{6}$ equals $-\frac{5\sqrt{3}}{4}$.
Also, at $t = \frac{\pi}{6}$, $x = 2\sqrt{3}$ and $y = \frac{5}{2}$. The equation for the tangent line is $y - \frac{5}{2} = -\frac{5\sqrt{3}}{4}(x - 2\sqrt{3})$, or $y = \left(-\frac{5\sqrt{3}}{4}\right)x + 10$.

6. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-4\sin t}$, which for $t = \frac{\pi}{6}$ equals $-\frac{5\sqrt{3}}{4}$.
Also, at $t = \frac{\pi}{6}$, $x = 2\sqrt{3}$ and $y = \frac{5}{2}$. The equation for the normal line is $y - \frac{5}{2} = \frac{4}{5\sqrt{3}}(x - 2\sqrt{3})$, or $y = \left(\frac{4\sqrt{3}}{15}\right)x + \frac{9}{10}$.

7. $\lim_{x\to 2} \frac{x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{x - 2}{(x + 2)(x - 2)} = \lim_{x\to 2} \frac{1}{x + 2} = \frac{1}{4}$

8. $y' = 3 - 2x$, and Length $= \int_0^2 \sqrt{1 + (3 - 2x)^2}\,dx$, which using NINT evaluates to $\approx 3.400$.

9. $x' = t\cos t + \sin t$, and $y' = -t\sin t + \cos t$, and
$$\text{Length} = \int_0^2 \sqrt{(t\cos t + \sin t)^2 + (-t\sin t + \cos t)^2}\,dt = \int_0^2 \sqrt{t^2 + 1}\,dt,$$
which using NINT evaluates to $\approx 2.958$.

10. $y = xe^x - e^x + C$ (use integration by parts), so
$2 = 0 - e^0 + C$ and $C = 3$:
$y = xe^x - e^x + 3$

### Section 10.3 Exercises

1. $[5 - (-1)]\mathbf{i} + (1 - 4)\mathbf{j} = 6\mathbf{i} - 3\mathbf{j}$

2. $(0 - 3)\mathbf{i} + [0 - (-4)]\mathbf{j} = -3\mathbf{i} + 4\mathbf{j}$

3. $\overrightarrow{AB} = [0 - (-3)]\mathbf{i} + (2 - 0)\mathbf{j} = 3\mathbf{i} + 2\mathbf{j}$ and
$\overrightarrow{CD} = (0 - 4)\mathbf{i} + (-3 - 0)\mathbf{j} = -4\mathbf{i} - 3\mathbf{j}$.

(a) $[3 + (-4)]\mathbf{i} + [2 + (-3)]\mathbf{j} = -\mathbf{i} - \mathbf{j}$

(b) $[3 - (-4)]\mathbf{i} + [2 - (-3)]\mathbf{j} = 7\mathbf{i} + 5\mathbf{j}$

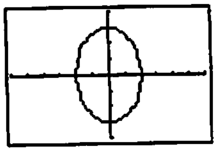
4. (a) $(5 + 3)\mathbf{i} + [(-2) + 4]\mathbf{j} = 8\mathbf{i} + 2\mathbf{j}$

(b) $(5 - 3)\mathbf{i} + [(-2) - 4]\mathbf{j} = 2\mathbf{i} - 6\mathbf{j}$

(c) $3(5)\mathbf{i} + 3(-2)\mathbf{j} = 15\mathbf{i} - 6\mathbf{j}$

(d) $[2(5)\mathbf{i} + 2(-2)\mathbf{j}] - [3(3)\mathbf{i} + 3(4)\mathbf{j}] = \mathbf{i} - 16\mathbf{j}$

**5. (a)**



[−6, 6] by [−4, 4]

**(b)** $\mathbf{v}(t) = \frac{d}{dt}(2\cos t)\mathbf{i} + \frac{d}{dt}(3\sin t)\mathbf{j}$

$= (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}(-2\sin t)\mathbf{i} + \frac{d}{dt}(3\cos t)\mathbf{j}$

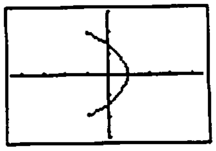
$= (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$

**(c)** $\mathbf{v}\left(\frac{\pi}{2}\right) = \langle -2, 0\rangle$; speed $= \sqrt{(-2)^2 + 0^2} = 2$,

direction $= \frac{1}{2}\langle -2, 0\rangle = \langle -1, 0\rangle$

**(d)** Velocity $= 2\langle -1, 0\rangle$

**6. (a)**



[−4.5, 4.5] by [−3, 3]

**(b)** $\mathbf{v}(t) = \frac{d}{dt}(\cos 2t)\mathbf{i} + \frac{d}{dt}(2\sin t)\mathbf{j}$

$= (-2\sin 2t)\mathbf{i} + (2\cos t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}(-2\sin 2t)\mathbf{i} + \frac{d}{dt}(2\cos t)\mathbf{j}$

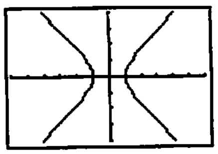
$= (-4\cos t)\mathbf{i} - (2\sin t)\mathbf{j}$

**(c)** $\mathbf{v}(0) = \langle 0, 2\rangle$; speed $= \sqrt{0^2 + 2^2} = 2$,

direction $= \frac{1}{2}\langle 0, 2\rangle = \langle 0, 1\rangle$

**(d)** Velocity $= 2\langle 0, 1\rangle$

**7. (a)**



[−6, 6] by [−4, 4]

**(b)** $\mathbf{v}(t) = \frac{d}{dt}(\sec t)\mathbf{i} + \frac{d}{dt}(\tan t)\mathbf{i} = (\sec t\tan t)\mathbf{i} + (\sec^2 t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}(\sec t\tan t)\mathbf{i} + \frac{d}{dt}(\sec^2 t)\mathbf{j}$

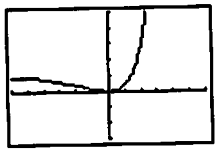
$= (\sec t\tan^2 t + \sec^3 t)\mathbf{i} + (2\sec^2 t\tan t)\mathbf{j}$

**(c)** $\mathbf{v}\left(\frac{\pi}{6}\right) = \left\langle \frac{2}{3}, \frac{4}{3}\right\rangle$; speed $= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \frac{2\sqrt{5}}{3}$,

direction $= \frac{3}{2\sqrt{5}}\left\langle \frac{2}{3}, \frac{4}{3}\right\rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

**(d)** Velocity $= \frac{2\sqrt{5}}{3}\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

**8. (a)**



[−6, 6] by [−3, 5]

**(b)** $\mathbf{v}(t) = \frac{d}{dt}(2\ln(t+1))\mathbf{i} + \frac{d}{dt}(t^2)\mathbf{j}$

$= \left(\frac{2}{t+1}\right)\mathbf{i} + (2t)\mathbf{j}$

$\mathbf{a}(t) = \frac{d}{dt}\left(\frac{2}{t+1}\right)\mathbf{i} + \frac{d}{dt}(2t)\mathbf{j} = \left(-\frac{2}{(t+1)^2}\right)\mathbf{i} + 2\mathbf{j}$

**(c)** $\mathbf{v}(1) = \langle 1, 2\rangle$; speed $= \sqrt{1^2 + 2^2} = \sqrt{5}$,

direction $= \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

**(d)** Velocity $= \sqrt{5}\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

**9.** $\mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j}$, $\mathbf{r}(0) = -\mathbf{j}$ and $\mathbf{v}(0) = \mathbf{i}$.

So the slope is zero (the velocity vector is horizontal).

**(a)** The horizontal line through $(0, -1)$: $y = -1$.

**(b)** The vertical line through $(0, -1)$: $x = 0$.

**10.** $\mathbf{v}(t) = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j}$,

$\mathbf{r}\left(\frac{\pi}{4}\right) = (\sqrt{2} - 3)\mathbf{i} + \left(\frac{3}{\sqrt{2}} + 1\right)\mathbf{j}$ and

$\mathbf{v}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(\frac{3}{\sqrt{2}}\right)\mathbf{j}$. So the slope is $\frac{3/\sqrt{2}}{-\sqrt{2}} = -\frac{3}{2}$.

**(a)** $y - \left(\frac{3}{\sqrt{2}} + 1\right) = -\frac{3}{2}[x - (\sqrt{2} - 3)]$ or

$y = -\frac{3}{2}x + \frac{6\sqrt{2} - 7}{2}$

**(b)** $y - \left(\frac{3}{\sqrt{2}} + 1\right) = \frac{2}{3}[x - (\sqrt{2} - 3)]$ or

$y = \frac{2}{3}x + \frac{5\sqrt{2} + 18}{6}$

**11.** $\left(\int_1^2 (6 - 6t)\,dt\right)\mathbf{i} + \left(\int_1^2 3\sqrt{t}\,dt\right)\mathbf{j}$

$= \left[6t - 3t^2\right]_1^2\mathbf{i} + \left[2t^{3/2}\right]_1^2\mathbf{j}$

$= -3\mathbf{i} + (4\sqrt{2} - 2)\mathbf{j}$

**12.** $\left(\int_{-\pi/4}^{\pi/4} \sin t\,dt\right)\mathbf{i} + \left(\int_{-\pi/4}^{\pi/4} (1 + \cos t)\,dt\right)\mathbf{j}$

$= \left[-\cos t\right]_{-\pi/4}^{\pi/4}\mathbf{i} + \left[t + \sin t\right]_{-\pi/4}^{\pi/4}\mathbf{j}$

$= \left(\sqrt{2} + \frac{\pi}{2}\right)\mathbf{j}$

13. $\left(\int \sec t \tan t\, dt\right)\mathbf{i} + \left(\int \tan t\, dt\right)\mathbf{j}$

$= (\sec t + C_1)\mathbf{i} + (\ln|\sec t| + C_2)\mathbf{j}$

$= (\sec t)\mathbf{i} + (\ln|\sec t|)\mathbf{j} + \mathbf{C}$

14. $\left(\int \frac{1}{t}\, dt\right)\mathbf{i} + \left(\int \frac{1}{5-t}\, dt\right)\mathbf{j}$

$= (\ln|t| + C_1)\mathbf{i} + (-\ln|5-t| + C_2)\mathbf{j}$

$= (\ln|t|)\mathbf{i} - (\ln|5-t|)\mathbf{j} + \mathbf{C}$

15. $\mathbf{r}(t) = (t+1)^{3/2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{C}$, and
$\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + \mathbf{C} = \mathbf{0}$, so $\mathbf{C} = -(\mathbf{i} - \mathbf{j}) = -\mathbf{i} + \mathbf{j}$
$\mathbf{r}(t) = ((t+1)^{3/2} - 1)\mathbf{i} - (e^{-t} - 1)\mathbf{j}$

16. $\mathbf{r}(t) = \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j} + \mathbf{C}$, and $\mathbf{r}(0) = \mathbf{C} = \mathbf{i} + \mathbf{j}$, so

$\mathbf{r}(t) = \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j}.$

17. $\frac{d\mathbf{r}}{dt} = (-32t)\mathbf{j} + \mathbf{C}_1$ and $\mathbf{r}(t) = (-16t^2)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$.

$\mathbf{r}(0) = \mathbf{C}_2 = 100\mathbf{i}$, and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$. So

$\mathbf{r}(t) = (-16t^2)\mathbf{j} + (8\mathbf{i} + 8\mathbf{j})t + 100\mathbf{i}$

$= (8t + 100)\mathbf{i} + (-16t^2 + 8t)\mathbf{j}.$

18. $\frac{d\mathbf{r}}{dt} = -t\mathbf{i} - t\mathbf{j} + \mathbf{C}_1$, and

$\mathbf{r}(t) = \left(-\frac{t^2}{2}\right)\mathbf{i} + \left(-\frac{t^2}{2}\right)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$

$\mathbf{r}(0) = \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j}$, and

$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1 = \mathbf{0}$, so

$\mathbf{r}(t) = \left(-\frac{t^2}{2}\right)\mathbf{i} + \left(-\frac{t^2}{2}\right)\mathbf{j} + (10\mathbf{i} + 10\mathbf{j})$

$= \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j}$

19. $\mathbf{v}(t) = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$.
Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(\sin t - \sin t \cos t) + (\sin t \cos t) = 0$
implies $\sin t = 0$, which is true for $t = 0, \pi$, or $2\pi$.

20. $\mathbf{v}(t) = (\cos t)\mathbf{i} + \mathbf{j}$, and $\mathbf{a}(t) = (-\sin t)\mathbf{i}$.

Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $-\sin t \cos t = 0$, which is true for

$t = \frac{k\pi}{2}$, $k$ any nonnegative integer.

21. $\mathbf{v}(t) = (-3\sin t)\mathbf{i} + (4\cos t)\mathbf{j}$, and

$\mathbf{a}(t) = (-3\cos t)\mathbf{i} + (-4\sin t)\mathbf{j}$.

Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(9\sin t \cos t) - (16\sin t \cos t) = 0$, is

true when $\sin t = 0$ or $\cos t = 0$, i.e., for

$t = \frac{k\pi}{2}$, $k$ any nonnegative integer.

22. $\mathbf{v}(t) = (-5\sin t)\mathbf{i} + (5\cos t)\mathbf{j}$, and

$\mathbf{a}(t) = (-5\cos t)\mathbf{i} + (-5\sin t)\mathbf{j}$.

Solve $\mathbf{v} \cdot \mathbf{a} = 0$: $(25\sin t \cos t) + (-25\sin t \cos t) = 0$,

which is true for all values of $t$.

23. $\mathbf{v}(t) = (-2\sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$\mathbf{a}(t) = (-2\cos t)\mathbf{i} + (-\sin t)\mathbf{j}$. So

$\mathbf{v}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(\frac{1}{\sqrt{2}}\right)\mathbf{j}$, and
$\mathbf{a}\left(\frac{\pi}{4}\right) = (-\sqrt{2})\mathbf{i} + \left(-\frac{1}{\sqrt{2}}\right)\mathbf{j}$.
Then $|\mathbf{v}| = |\mathbf{a}| = \sqrt{\frac{5}{2}}$,

$\mathbf{v} \cdot \mathbf{a} = \frac{3}{2}$, and

$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}||\mathbf{a}|}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$.

24. $\mathbf{v}(t) = 3\mathbf{i} + (2t)\mathbf{j}$, and $\mathbf{a}(t) = 2\mathbf{j}$. So $\mathbf{v}(0) = 3\mathbf{i}$, and
$\mathbf{a}(0) = 2\mathbf{j}$. These are perpendicular, i.e., the angle between them measures 90°.

25. (a) Both components are continuous at $t = 3$, so the limit

is $3\mathbf{i} + \left(\frac{3^2 - 9}{3^2 + 3(3)}\right)\mathbf{j} = 3\mathbf{i}$.

(b) Continuous so long as $t^2 + 3t \neq 0$, i.e., $t \neq 0, -3$

(c) Discontinuous when $t^2 + 3t = 0$, i.e., $t = 0$ or $-3$

26. (a) Use L'Hôpital's Rule for the i-component:

$\lim_{t\to 0}\left(\frac{\sin 2t}{t}\right)\mathbf{i} + \lim_{t\to 0}(\ln(t+1))\mathbf{j}$

$= \lim_{t\to 0}\left(\frac{2\cos 2t}{1}\right)\mathbf{i} + \lim_{t\to 0}(\ln(t+1))\mathbf{j}$

$= 2\mathbf{i} + 0\mathbf{j} = 2\mathbf{i}.$

(b) Continuous so long as $t \neq 0$ and $t + 1 > 0$, i.e., $(-1, 0) \cup (0, \infty)$.

(c) Discontinuous when $t = 0$ or $t + 1 \leq 0$, i.e., $(-\infty -1] \cup \{0\}$.

27. $\mathbf{v}(t) = (\sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$, i.e.,

$\frac{dx}{dt} = \sin t$, and $\frac{dy}{dt} = 1 - \cos t$

$\text{Distance} = \int_0^{2\pi/3} \sqrt{(\sin t)^2 + (1 - \cos t)^2}\, dt$

$= \int_0^{2\pi/3} \sqrt{2 - 2\cos t}\, dt$

$= \int_0^{2\pi/3} 2\sin\left(\frac{t}{2}\right) dt$

$= \left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi/3} = 2$

**28.** (a) $\mathbf{r}(0) = \left(\frac{1}{4}e^0 - 0\right)\mathbf{i} + (e^0)\mathbf{j}$

$= \frac{1}{4}\mathbf{i} + \mathbf{j}$,

$\mathbf{r}(2) = \left(\frac{1}{4}e^8 - 2\right)\mathbf{i} + (e^4)\mathbf{j}$

Initial $= \left(\frac{1}{4}, 1\right)$, terminal $= \left(\frac{1}{4}e^8 - 2, e^4\right)$

(b) $\mathbf{v}(t) = (e^{4t} - 1)\mathbf{i} + (2e^{2t})\mathbf{j}$; $\frac{dx}{dt} = e^{4t} - 1$, and

$\frac{dy}{dt} = 2e^{2t}$.

$$\text{Length} = \int_0^2 \sqrt{(e^{4t}-1)^2 + (2e^{2t})^2}\,dt = \int_0^2 \sqrt{(e^{4t}+1)^2}\,dt = \int_0^2 (e^{4t}+1)\,dt = \left[\frac{1}{4}e^{4t} + t\right]_0^2 = \frac{e^8+7}{4} \approx 746.989$$

(c) $$\int_0^2 2\pi\left(\frac{1}{4}e^{4t} - t\right)\sqrt{(e^{4t}-1)^2 + (2e^{2t})^2}\,dt$$
$$= 2\pi\int_0^2 \left(\frac{1}{4}e^{4t} - t\right)(e^{4t}+1)\,dt$$
$$= 2\pi\int_0^2 \left(\frac{1}{4}e^{8t} + \frac{1}{4}e^{4t} - te^{4t} - t\right)dt$$
$$= 2\pi\left[\frac{1}{32}e^{8t} + \frac{1}{16}e^{4t} - \frac{1}{16}(4t-1)e^{4t} - \frac{1}{2}t^2\right]_0^2$$
$$= \pi\left(\frac{e^{16} - 12e^8 - 69}{16}\right) \approx 1{,}737{,}746.456$$

**29.** (a) $\mathbf{v}(t) = (\cos t)\mathbf{i} - (2\sin 2t)\mathbf{j}$

(b) $\mathbf{v}(t) = 0$ when both $\cos t = 0$ and $\sin 2t = 0$. $\cos t = 0$ at $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$; $\sin 2t = 0$ at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and $2\pi$. So $\mathbf{v}(t) = 0$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$.

(c) $x = \sin t$, $y = \cos 2t$. Relate the two using the identity $\cos 2u = 1 - 2\sin^2 u$: $y = 1 - 2x^2$, where as $t$ ranges over all possible values, $-1 \le x \le 1$. When $t$ increases from 0 to $2\pi$, the particle starts at (0, 1), goes to (1, −1), then goes to (−1, −1), and then goes to (0, 1), tracing the curve twice.

**30.** (a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-12}{6t^2-6t} = \frac{t^2-4}{2t^2-2t}$

(b) Horizontal tangents: $t^2 - 4 = 0$ for $t = \pm 2$.
Vertical tangents: $2t^2 - 2t = 0$ for $t = 0, 1$.
Plugging the $t$-values into $x = 2t^3 - 3t^2$ and $y = t^3 - 12t$ produces the $x$- and $y$-coordinates of the critical points.
$t = -2$: horizontal tangent at (−28, 16)
$t = 0$: vertical tangent at (0, 0)
$t = 1$: vertical tangent at (−1, −11)
$t = 2$: horizontal tangent at (4, −16)

**31.** $\mathbf{a}(t) = 3\mathbf{i} - \mathbf{j}$, so $\mathbf{v}(t) = (3t)\mathbf{i} - t\mathbf{j} + \mathbf{C}_1$ and

$\mathbf{r}(t) = \left(\frac{3}{2}t^2\right)\mathbf{i} - \left(\frac{1}{2}t^2\right)\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$. $\mathbf{r}(0) = \mathbf{C}_2 = \mathbf{i} + 2\mathbf{j}$,

and since $\mathbf{v}(0)$ must point directly from (1, 2) toward (4, 1) with magnitude 2,

$$\mathbf{v}(0) = \mathbf{C}_1 = 2\left(\frac{(4-1)\mathbf{i} + (1-2)\mathbf{j}}{\sqrt{(4-1)^2 + (1-2)^2}}\right) = \frac{6}{\sqrt{10}}\mathbf{i} - \frac{2}{\sqrt{10}}\mathbf{j} = \frac{3\sqrt{10}}{5}\mathbf{i} - \frac{\sqrt{10}}{5}\mathbf{j}$$

So $\mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{3\sqrt{10}}{5}t + 1\right)\mathbf{i} + \left(-\frac{1}{2}t^2 - \frac{\sqrt{10}}{5}t + 2\right)\mathbf{j}$.

**32.** (a) $\frac{dx}{dt} = 1 - \frac{2}{t^2} = 0$ when $t = \sqrt{2}$. That corresponds to point $\left(\sqrt{2} + \frac{2}{\sqrt{2}}, 3(\sqrt{2})^2\right) = (2\sqrt{2}, 6)$.

(b) $\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{6t}{1 - 2/t^2}$, which for $t = 1$ equals −6.

(c) When $y = 12$, $t = 2$.
$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{(1-2/t^2)6 - (4/t^3)6t}{(1-2/t^2)^3}$,
which for $t = 2$ equals −24.

**33.** (a) The $\mathbf{j}$-component is zero at $t = 0$ and $t = 160$: 160 seconds.

(b) $-\frac{3}{64}(40)(40 - 160) = 225$ m

(c) $\frac{d}{dt}\left[-\frac{3}{64}t(t - 160)\right] = -\frac{3}{32}t + \frac{15}{2}$, which for $t = 40$ equals $\frac{15}{4}$ m per second.

(d) $v(t) = -\frac{3}{32}t + \frac{15}{2}$ equals 0 at $t = 80$ seconds (and is negative after that time).

**34.** (a) Solve $t - 3 = \frac{3t}{2} - 4$: $t = 2$. Then check that $(t-3)^2 = \frac{3t}{2} - 2$ for $t = 2$: it does.

(b) First particle: $\mathbf{v}_1(t) = \mathbf{i} + 2(t-3)\mathbf{j}$, so $\mathbf{v}_1(2) = \mathbf{i} - 2\mathbf{j}$ and the direction unit vector $\mathbf{v}_1$ is $\left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle$.
Second particle: $\mathbf{v}_2(t) = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$, which is constant, and the direction unit vector is $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$.

**35.** (a) Referring to the figure, look at the circular arc from the point where $t = 0$ to the point "$m$". On one hand, this arc has length given by $r_0\theta$, but it also has length given by $vt$. Setting those two quantities equal gives the result.

(b) $\mathbf{v}(t) = \left(-v \sin \frac{vt}{r_0}\right)\mathbf{i} + \left(v \cos \frac{vt}{r_0}\right)\mathbf{j}$, and

$$\mathbf{a}(t) = \left(-\frac{v^2}{r_0} \cos \frac{vt}{r_0}\right)\mathbf{i} + \left(-\frac{v^2}{r_0} \sin \frac{vt}{r_0}\right)\mathbf{j} = -\frac{v^2}{r_0}\left[\left(\cos \frac{vt}{r_0}\right)\mathbf{i} + \left(\sin \frac{vt}{r_0}\right)\mathbf{j}\right]$$

(c) From part (b) above, $\mathbf{a}(t) = -\left(\frac{v}{r_0}\right)^2 \mathbf{r}(t)$.

So, by Newton's second law, $\mathbf{F} = -m\left(\frac{v}{r_0}\right)^2 \mathbf{r}$.

Substituting for $\mathbf{F}$ in the law of gravitation gives the result.

(d) Set $\frac{vT}{r_0} = 2\pi$ and solve for $vT$.

(e) Substitute $\frac{2\pi r_0}{T}$ for $v$ in $v^2 = \frac{GM}{r_0}$ and solve for $T^2$:

$$\left(\frac{2\pi r_0}{T}\right)^2 = \frac{GM}{r_0}$$

$$\frac{4\pi^2 r_0^2}{T^2} = \frac{GM}{r_0}$$

$$\frac{1}{T^2} = \frac{GM}{4\pi^2 r_0^3}$$

$$T^2 = \frac{4\pi^2}{GM} r_0^3$$

**36.** Solve both equations for $t$: $t = e^x - 1$ and $t = \sqrt{y + 1}$. Now eliminate the $t$ and solve for $y$: $e^x - 1 = \sqrt{y + 1}$, $y = (e^x - 1)^2 - 1$, or $y = e^{2x} - 2e^x$, where $t \geq 0$ so $x \geq 0$.

**37.** (a) Apply Corollary 3 to each component separately. If the components all differ by scalar constants, the difference vector is a constant vector.

(b) Follows immediately from (a) since any two anti-derivatives of $\mathbf{r}(t)$ must have identical derivatives, namely $\mathbf{r}(t)$.

**38.** $\frac{d}{dt}|\mathbf{v}|^2 = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v}' \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}' = 2\mathbf{v} \cdot \mathbf{v}' = 0.$

Therefore, $|\mathbf{v}|$ is constant.

**39.** Let $\mathbf{C} = \langle C_1, C_2 \rangle$. $\frac{d\mathbf{C}}{dt} = \left\langle \frac{dC_1}{dt}, \frac{dC_2}{dt} \right\rangle = \langle 0, 0 \rangle$.

**40.** (a) Suppose $\mathbf{u} = \langle u_1(t), u_2(t) \rangle$.

$$\frac{d}{dt}(c\mathbf{u}) = \frac{d}{dt}\langle cu_1(t), cu_2(t) \rangle = \left\langle \frac{d}{dt}(cu_1(t)), \frac{d}{dt}(cu_2(t)) \right\rangle = \left\langle c\frac{du_1}{dt}, c\frac{du_2}{dt} \right\rangle = c\left\langle \frac{du_1}{dt}, \frac{du_2}{dt} \right\rangle = c\frac{d\mathbf{u}}{dt}$$

(b)
$$\frac{d}{dt}(f\mathbf{u}) = \frac{d}{dt}\langle fu_1, fu_2 \rangle = \langle fu_1' + f'u_1, fu_2' + f'u_2 \rangle = \langle fu_1', fu_2' \rangle + \langle f'u_1, f'u_2 \rangle = f\mathbf{u}' + f'\mathbf{u}$$

**41.** $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$

(a)
$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d}{dt}(\langle u_1 + v_1, u_2 + v_2 \rangle) = \left\langle \frac{d}{dt}(u_1 + v_1), \frac{d}{dt}(u_2 + v_2) \right\rangle = \langle u_1' + v_1', u_2' + v_2' \rangle = \langle u_1', u_2' \rangle + \langle v_1', v_2' \rangle = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$$

(b)
$$\frac{d}{dt}(\mathbf{u} - \mathbf{v}) = \frac{d}{dt}(\langle u_1 - v_1, u_2 - v_2 \rangle) = \left\langle \frac{d}{dt}(u_1 - v_1), \frac{d}{dt}(u_2 - v_2) \right\rangle = \langle u_1' - v_1', u_2' - v_2' \rangle = \langle u_1', u_2' \rangle - \langle v_1', v_2' \rangle = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$$

**42.** $\frac{d\mathbf{r}}{dt} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j}$

$$\left(\frac{d\mathbf{r}}{dt}\right)\left(\frac{dt}{ds}\right) = \left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j}\right)\left(\frac{dt}{ds}\right) = \left(\frac{df}{dt} \cdot \frac{dt}{ds}\right)\mathbf{i} + \left(\frac{dg}{dt} \cdot \frac{dt}{ds}\right)\mathbf{j} = \frac{df}{ds}\mathbf{i} + \frac{dg}{ds}\mathbf{j} = \frac{d\mathbf{r}}{ds}$$

**43.** $f(t)$ and $g(t)$ differentiable at $c \Rightarrow f(t)$ and $g(t)$ continuous at $c \Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is continuous at $c$.

**44.** (a) Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.

$$\begin{aligned}\int_a^b k\mathbf{r}(t)\,dt &= \int_a^b \langle kx(t), ky(t)\rangle\,dt = \left\langle \int_a^b kx(t)\,dt, \int_a^b ky(t)\,dt\right\rangle \\ &= \left\langle k\int_a^b x(t)\,dt, k\int_a^b y(t)\,dt\right\rangle \\ &= k\left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt\right\rangle = k\int_a^b \langle x(t), y(t)\rangle\,dt \\ &= k\int_a^b \mathbf{r}(t)\,dt\end{aligned}$$

(b) Let $\mathbf{r}_1(t) = \langle x_1(t), y_1(t)\rangle$ and $\mathbf{r}_2(t) = \langle x_2(t), y_2(t)\rangle$.

$$\int_a^b (\mathbf{r}_1(t) \pm \mathbf{r}_2(t))\,dt = \int_a^b (\langle x_1(t), y_1(t)\rangle \pm \langle x_2(t), y_2(t)\rangle)\,dt$$

$$= \int_a^b \langle x_1(t) \pm x_2(t), y_1(t) \pm y_2(t)\rangle\,dt$$

$$= \left\langle \int_a^b (x_1(t) \pm x_2(t))\,dt, \int_a^b (y_1(t) \pm y_2(t))\,dt\right\rangle$$

$$= \left\langle \int_a^b x_1(t)\,dt \pm \int_a^b x_2(t)\,dt, \int_a^b y_1(t)\,dt \pm \int_a^b y_2(t)\,dt\right\rangle$$

$$= \left\langle \int_a^b x_1(t)\,dt, \int_a^b y_1(t)\,dt\right\rangle \pm \left\langle \int_a^b x_2(t)\,dt, \int_a^b y_2(t)\,dt\right\rangle$$

$$= \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$$

(c) Let $\mathbf{C} = \langle C_1, C_2\rangle$, $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.

$$\begin{aligned}\int_a^b \mathbf{C}\cdot\mathbf{r}(t)\,dt &= \int_a^b (C_1x(t) + C_2y(t))\,dt \\ &= C_1\int_a^b x(t)\,dt + C_2\int_a^b y(t)\,dt \\ &= \langle C_1, C_2\rangle \cdot \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt\right\rangle \\ &= \mathbf{C}\cdot\int_a^b \mathbf{r}(t)\,dt\end{aligned}$$

**45.** (a) Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$. Then

$$\begin{aligned}\frac{d}{dt}\int_a^t \mathbf{r}(q)\,dq &= \frac{d}{dt}\int_a^t [f(q)\mathbf{i} + g(q)\mathbf{j}]\,dq \\ &= \frac{d}{dt}\left[\left(\int_a^t f(q)\,dq\right)\mathbf{i} + \left(\int_a^t g(q)\,dq\right)\mathbf{j}\right] \\ &= \left(\frac{d}{dt}\int_a^t f(q)\,dq\right)\mathbf{i} + \left(\frac{d}{dt}\int_a^t g(q)\,dq\right)\mathbf{j} \\ &= f(t)\mathbf{i} + g(t)\mathbf{j} = \mathbf{r}(t).\end{aligned}$$

(b) Let $S(t) = \int_a^t \mathbf{r}(q)\,dq$. Then part (a) shows that $\mathbf{S}(t)$ is an antiderivative of $\mathbf{r}(t)$. Let $\mathbf{R}(t)$ be any antiderivative of $\mathbf{r}(t)$. Then according to 37(b), $\mathbf{S}(t) = \mathbf{R}(t) + \mathbf{C}$.

Letting $t = a$, we have $0 = \mathbf{S}(a) = \mathbf{R}(a) + \mathbf{C}$.

Therefore, $\mathbf{C} = -\mathbf{R}(a)$ and $\mathbf{S}(t) = \mathbf{R}(t) - \mathbf{R}(a)$.

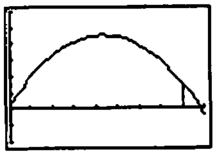
The result follows by letting $t = b$.

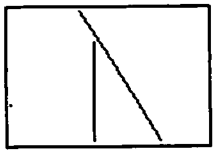
## Section 10.4 Modeling Projectile Motion

(pp. 539–552)

### Exploration 1 Hitting a Home Run

1. The graphs of the parametric equations $x = (152\cos 20° - 8.8)t$, $y = 3 + (152\sin 20°)t - 16t^2$ and the fence are shown in the window [0, 450] by [−20, 60]. The fence was obtained using the line command "Line(". You can zoom in as shown in the second figure to see that the ball does just clear the fence.



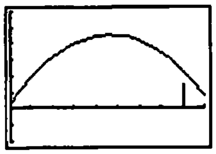


You can also use algebraic methods to show that $t \approx 2.984$ when $x = 400$, and that $y \approx 15.647$ for this value of $t$.

2.

| angle (degrees) | 25 | 30 | 45 |
|---|---|---|---|
| range (ft) | ≈523.707 | ≈588.279 | ≈665.629 |
| flight time (sec) | ≈4.061 | ≈4.789 | ≈6.745 |

3. Using the same window of part (1) we can see that the ball clears the fence.



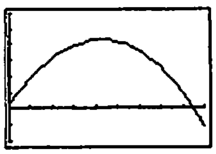
4.

| angle (degrees) | 25 | 30 | 45 |
|---|---|---|---|
| range (ft) | ≈559.444 | ≈630.424 | ≈724.988 |
| flight time (sec) | ≈4.061 | ≈4.789 | ≈6.745 |

### Exploration 2 Hitting a Baseball

1. $x = \frac{152}{0.05}(1 - e^{-0.05t})\cos 20°$

$$y = 3 + \frac{152}{0.05}(1 - e^{-0.05t})\sin 20° + \frac{32}{0.05^2}(1 - 0.05t - e^{-0.05t})$$



[0, 450] by [−20, 60]

2. The ball reaches a maximum height of about 43.07 ft when $t$ is about 1.56 sec.
3. The range is about 425.47 ft and the flight time is about 3.23 sec.

## Quick Review 10.4

1. $\langle 50\cos 25°, 50\sin 25°\rangle \approx \langle 45.315, 21.131\rangle$
2. $\langle 80\cos 120°, 80\sin 120°\rangle = \langle -40, 40\sqrt{3}\rangle$
3. To find the $x$-intercepts, solve $2x^2 + 11x - 40 = 0$ using the quadratic formula: $x = \frac{-11 \pm \sqrt{11^2 - 4(2)(-40)}}{2(2)}$ $= \frac{5}{2}$ or $-8$. The $x$-intercepts are $\left(\frac{5}{2}, 0\right)$ and $(-8, 0)$. For the $y$-intercept, find $f(0) = 2(0)^2 + 11(0) - 40 = -40$. The $y$-intercept is $(0, -40)$.
4. At the vertex, $f'(x) = 4x + 11 = 0$ and $x = -\frac{11}{4}$. Then the vertex is $\left(-\frac{11}{4}, 2\left(-\frac{11}{4}\right)^2 + 11\left(-\frac{11}{4}\right) - 40\right)$ $= \left(-\frac{11}{4}, -\frac{441}{8}\right)$.
5. To find the $x$-intercepts, solve $20x - x^2 = 0$: $x = 0$ or $20$. The $x$-intercepts are $(0, 0)$ and $(20, 0)$. For the $y$-intercept, find $y(0)$: it is already known to be 0. So the $y$-intercept is $(0, 0)$.
6. At the vertex, $g'(x) = 20 - 2x = 0$ and $x = 10$. Then the vertex is $(10, 20(10) - 10^2) = (10, 100)$.
7. $y = -\cos x + C$. $y\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) + C = C = 2$, so $y = -\cos x + 2$.
8. $y' = t^2 + C_1$ and $y = \frac{1}{3}t^3 + C_1 t + C_2$

   $y'(-1) = (-1)^2 + C_1 = 1 + C_1 = 4$, so $C_1 = 3$.

   $y(-1) = \frac{1}{3}(-1)^3 + 3(-1) + C_2 = -\frac{10}{3} + C_2 = 5$, so

   $C_2 = \frac{25}{3}$

   $y = \frac{t^3}{3} + 3t + \frac{25}{3}$
9. $\int \frac{dy}{16 - y} = \int dt$

   $-\ln|16 - y| = t + C$

   $16 - y = ke^{-t}$

   $y(0) = 16 - k = 20$ so $k = -4$

   $y = 16 + 4e^{-t}$
10. $\int \frac{dy}{4 - 2y} = \int x\,dx$

    $-\frac{1}{2}\ln|4 - 2y| = \frac{1}{2}x^2 + C$

    $4 - 2y = ke^{-x^2}$

    $y = 2 - \frac{k}{2}e^{-x^2}$

    $y(0) = 2 - \frac{k}{2} = 1$ so $k = 2$

    $y = 2 - e^{-x^2}$

## Section 10.4 Exercises

1. Solve $v_x t = (840\cos 60°)t = 21{,}000$ for $t$: $t = 50$ seconds.
2. Use $R = \frac{v_0^2}{g}\sin 2\alpha$; solve $24{,}500 = \frac{v_0^2}{9.8}\sin 90°$ for $v_0$: $v_0 = 490$ m/sec.
3. (a) $t = \frac{2v_0 \sin\alpha}{g} = \frac{2(500)\sin 45°}{9.8} \approx 72.154$ seconds;

   $R = \frac{v_0^2}{g}\sin 2\alpha = \frac{500^2}{9.8}\sin 90° \approx 25{,}510$ m

   $= 25.510$ km downrange

   (b) $y = -\left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2 + (\tan\alpha)x$

   $= -\left(\frac{9.8}{2(500)^2\cos^2 45°}\right)5000^2 + (\tan 45°)5000 = 4020$ m

   (c) $y_{max} = \frac{(v_0\sin\alpha)^2}{2g} = \frac{(500\sin 45°)^2}{2(9.8)} \approx 6377.551$ m
4. With the origin at the launch point

   (so the ground is $t = 2$ when $y = -32$), use

   $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$.

   $-32 = (32\sin 30°)t - \frac{1}{2}(32)t^2$

   $-2 = t - t^2$

   $t = 2$ seconds

   Then $x = (v_0\cos\alpha)t = (32\cos 30°)2 = 32\sqrt{3} \approx 55.426$ feet away (horizontally).
5. Use $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 + 6.5$.

   $16t^2 - 22\sqrt{2}t - 6.5 = 0$

   $t = \frac{11\sqrt{2} + \sqrt{346}}{16} \approx 2.135$ seconds (by the quadratic formula). Substitute that into $x = (v_0\cos\alpha)t$ $= (44\sin 45°)t$ to obtain $x \approx 66.4206$. 66.421 feet from the stopboard.
6. With the origin at the launch point, use

   $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$. $-6.5 = (44\sin 40°)t - 16t^2$

   $t = \frac{(44\sin 40°) + \sqrt{(44\sin 40°)^2 + 416}}{32} \approx 1.974$ sec

   At $t \approx 1.974$, $x = (44\cos 40°)t \approx 66.5193$ ft

   Thus the shot would have gone $\approx 0.0987$ feet $\approx 1.18$ inches farther.

7. (a) Use $R = \frac{v_0^2}{g}\sin 2\alpha$; solve $10 = \frac{v_0^2}{9.8}\sin 90°$ for $v_0$: $v_0 = 7\sqrt{2} \approx 9.899$ m/sec.

(b) Solve $6 = \frac{(7\sqrt{2})^2}{9.8}\sin 2\alpha$ for $\alpha$: $\sin 2\alpha = 0.6$, so $2\alpha = \sin^{-1} 0.6 \approx 36.870°$ and $\alpha \approx 18.435°$ or $2\alpha = 180° - \sin^{-1} 0.6 \approx 143.130$ and $\alpha \approx 71.565°$.

8. $t = \frac{40 \times 10^{-2}\text{ m}}{5 \times 10^6\text{ m/sec}} = 8 \times 10^{-8}$ sec. Then $y$ (taking down as positive) is $\frac{1}{2}gt^2 \approx \frac{1}{2}(9.8)(8 \times 10^{-8})^2 = 3.136 \times 10^{-14}$ meters or $3.136 \times 10^{-12}$ cm.

9. $R = \frac{v_0^2}{g}\sin 2\alpha$

$(248.8 \text{ yd})(3 \text{ ft/yd}) = \frac{v_0^2}{32 \text{ ft/sec}^2}\sin 18°$

$v_0 \approx 278.016$ ft/sec or $\approx 189.556$ mph.

10. $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 200 = \frac{\left(\frac{80\sqrt{10}}{3}\right)^2}{32}\sin 2\alpha \Rightarrow \sin 2\alpha = 0.9.$

Taking the smaller of the two possible angles, $\alpha = \frac{1}{2}\sin^{-1} 0.9 \approx 32.079°$. Then

$y_{max} \approx \frac{\left(\frac{80\sqrt{10}}{3}\right)^2 \sin^2 32.079}{2(32)} \approx 31.339$, which is well below the ceiling height.

11. No. For $\alpha = 30°$, $v_0 = 90$ ft/sec, and $x = 135$ ft, $y = -\left(\frac{32}{2v_0^2\cos^2\alpha}\right)x^2 + (\tan\alpha)x$ evaluates to $\approx 29.942$ feet above the ground, which is not quite high enough.

12. Use $y = -\left(\frac{32}{2(116)^2\cos^2 45°}\right)x^2 + (\tan 45°)x$ $= -\frac{2}{841}x^2 + x$. Set $y = 45$, then solve for $x$ using the quadratic formula and taking the larger of the two values:

$x = \frac{1 + \sqrt{\frac{481}{841}}}{\frac{4}{841}} \approx 369.255$ ft, which is $\approx 0.255$ ft $\approx 3.059$ inches beyond the pin.

13. (a) With the origin at the launch point, use $y = -\left(\frac{32}{2v_0^2\cos^2 20°}\right)x^2 + (\tan 20°)x$. Set $x = 315$ and $y = 37 - 3 = 34$, then solve to find $v_0 = \frac{1260}{\cos 20°\sqrt{315\tan 20° - 34}} \approx 149.307$ ft/sec.

(b) Solve $v_0(\cos 20°)t \approx 149.307(\cos 20°)t = 315$ to find $t \approx 2.245$ seconds.

14. In the formula for range, $\sin 2\alpha = \sin 2(90 - \alpha)$.

15. Use $R = \frac{v_0^2}{g}\sin 2\alpha$: $\sin 2\alpha = \frac{(9.8)(16{,}000)}{400^2} = 0.98$; $\alpha = \frac{\sin^{-1} 0.98}{2} \approx 39.261°$ or $\alpha = 90 - \frac{\sin^{-1} 0.98}{2}$ $\approx 50.739°$.

16. (a) Substitute $2v_0$ for $v_0$ in the formula for range.

(b) To increase the range (and height) by a factor of 2, increase $v_0$ by a factor of $\sqrt{2} \approx 1.41$. That is an increase of $\approx 41\%$.

17. With the origin at the launch point, $y = -\left(\frac{32}{2v_0^2\cos^2 40°}\right)x^2 + (\tan 40°)x$. Setting $x = 73\frac{5}{6}$ and $y = -6.5$ and solving for $v_0$ yields $v_0 \approx 46.597$ ft/sec.

18. $y(t) = v_0(\sin\alpha)t - \frac{1}{2}gt^2$, and we know the maximum height is $\frac{(v_0\sin\alpha)^2}{2g}$ and it occurs when $t = \frac{v_0\sin\alpha}{g}$. Substituting $t = \frac{v_0\sin\alpha}{2g}$ into the equation for $y(t)$ gives a height of $\frac{3(v_0\sin\alpha)^2}{8g}$, which is three-fourths of the maximum height.

19. Integrating, $\frac{d}{dt}\mathbf{r}(t) = c_1\mathbf{i} + (-gt + c_2)\mathbf{j}$. The initial condition on the velocity gives $c_1 = v_0\cos\alpha$ and $c_2 = v_0\sin\alpha$. Integrating again, $\mathbf{r}(t) = ((v_0\cos\alpha)t + c_3)\mathbf{i} + \left(\frac{1}{2}gt^2 + (v_0\sin\alpha)t + c_4\right)\mathbf{j}$. The initial condition on the position gives $c_3 = x_0$ and $c_4 = y_0$.

20. With the origin at the launch point, $y_{max} = 68$ ft. Then $v_0 = \frac{\sqrt{2y_{max}g}}{\sin\alpha} \approx v_0 = \frac{\sqrt{2(68)(32)}}{\sin 56.505°} \approx 79.107$ ft/sec.

21. The horizontal distance is 30 yd − 6 ft = 84 ft. Then $84 = (v_0\cos\alpha)t$, where $\alpha = \tan^{-1}\left(\frac{68}{45}\right) \approx 56.5°$ and $v_0 = \frac{16\sqrt{17}}{\sin\alpha}$ (from Exercise 20). So $t = \frac{84}{v_0\cos\alpha}$

$= \frac{84\tan\alpha}{16\sqrt{17}} = \frac{21\left(\frac{68}{45}\right)}{4\sqrt{17}} = \frac{119}{15\sqrt{17}} \approx 1.924$ seconds. Then

$y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$

$= (16\sqrt{17})\left(\frac{119}{15\sqrt{17}}\right) - \frac{1}{2}(32)\left(\frac{119}{15\sqrt{17}}\right)^2 \approx 67.698$ ft.

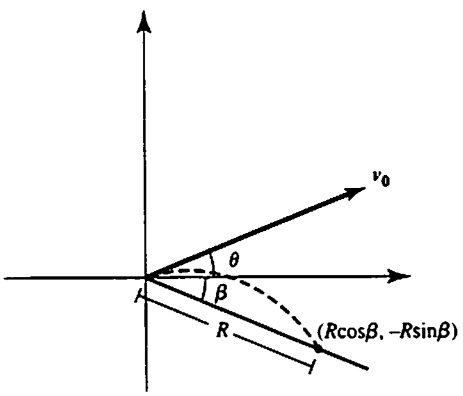
The height above the ground is 6 ft more than that, $\approx 73.698$, and the height above the rim is about $73.698 - 70 = 3.698$ feet.

22. The projectile rises straight up and then falls straight down, returning to the firing point.

**23.** Angle is $\alpha \approx 62°$ (measurements may vary slightly). For flight time $t = \frac{2v_0 \sin \alpha}{g} = 1$ sec, $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} = \frac{1}{8}gt^2$
$= \frac{1}{8}(32)(1)^2 = 4$ ft (independent of the measured angle).
$v_0 = \frac{gt}{2 \sin \alpha}$, so speed of engine $= v_0 \cos \alpha = \frac{gt}{2 \tan \alpha}$
$\approx \frac{32(1)}{2 \tan (62°)} \approx 8.507$ ft/sec (changes with the angle).

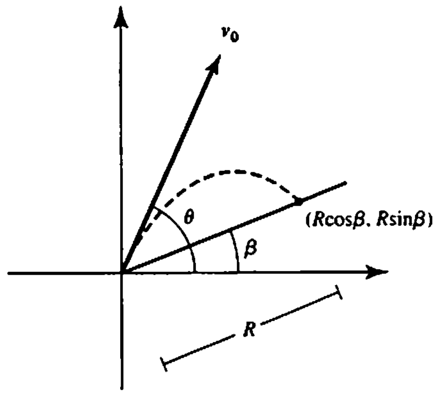
**24.** The height of $A$ is given by $y_A = (v \sin \alpha)t - \frac{1}{2}gt^2$ and the height of $B$ is given by $y_B = R \tan \alpha - \frac{1}{2}gt^2$. The second terms in $y_A$ and $y_B$ $\left(-\frac{1}{2}gt^2\right)$ are equal for any value of $t$. But $A$ moves $R$ units horizontally to $B$'s line of fall in $\frac{R}{v \cos \alpha}$ time units, and the first terms in $y_A$ and $y_B$ are also equal at that time: $(v \sin \alpha)\left(\frac{R}{v \cos \alpha}\right) = R \tan \alpha$. Therefore, $A$ and $B$ will always be at the same height when $A$ reaches $B$'s line of fall.

**25.** **(a)**




$x = (v_0 \cos \theta)t$
$y = (v_0 \sin \theta)t - \frac{1}{2}gt^2$
$x = R \cos \beta \Rightarrow R \cos \beta = (v_0 \cos \theta)t$
$\Rightarrow t = \frac{R \cos \beta}{v_0 \cos \theta}$. Then $y = -R \sin \beta$
$\Rightarrow -R \sin \beta = \frac{(v_0 \sin \theta) R \cos \beta}{v_0 \cos \theta} - \frac{g}{2} \frac{R^2 \cos^2 \beta}{v_0^2 \cos^2 \theta}$
$\Rightarrow R = \frac{2v_0^2}{g \cos^2 \beta} \cos \theta \sin (\theta + \beta)$.
Let $f(\theta) = \cos \theta \sin (\theta + \beta)$.
$f'(\theta) = \cos \theta \cos (\theta + \beta) - \sin \theta \sin (\theta + \beta)$
$f'(\theta) = 0 \Rightarrow \tan \theta \tan (\theta + \beta) = 1$
$\Rightarrow \tan \theta = \cot (\theta + \beta)$
$\Rightarrow \theta + \beta = 90° - \theta$. Note that $f''(\theta) < 0$, so $R$ is maximum when $\alpha = \theta + \beta = 90° - \theta$. Thus the initial velocity bisects angle $AOR$.

**(b)**




$R = \frac{2v_0^2}{g \cos^2 \beta} \cos \theta \sin (\theta - \beta)$ is maximum when $\tan \theta = \cot (\theta - \beta)$,
so $\theta - \beta = 90° - \theta$.
The initial velocity vector bisects the angle between the hill and the vertical for max range.

**26.** **(a)** $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where
$x(t) = (145 \cos 23° - 14)t$ and
$y(t) = 2.5 + (145 \sin 23°)t - 16t^2$.

**(b)** $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} + 2.5 = \frac{(145 \sin 23°)^2}{64} + 2.5$
$\approx 52.655$ feet, which is reached at $t = \frac{v_0 \sin \alpha}{g}$
$= \frac{145 \sin 23°}{32} \approx 1.771$ seconds.

**(c)** For the time, solve
$y = 2.5 + (145 \sin 23°)t - 16t^2 = 0$ for $t$, using the quadratic formula:
$t = \frac{145 \sin 23° + \sqrt{(145 \sin 23°)^2 + 160}}{32} \approx 3.585$ sec.
Then the range at $t \approx 3.585$ is about
$x = (145 \cos 23° - 14)(3.585) \approx 428.262$ feet.

**(d)** For the time, solve $y = 2.5 + (145 \sin 23°)t - 16t^2$
$= 20$ for $t$, using the quadratic formula:
$t = \frac{145 \sin 23° \pm \sqrt{(145 \sin 23°)^2 - 1120}}{32} \approx 0.342$ and 3.199 seconds. At those times the ball is about
$x(0.342) = (145 \cos 23° - 14)(0.342) \approx 40.847$ feet
and $x(3.199) = (145 \cos 23° - 14)(3.199) \approx 382.208$ feet from home plate.

**(e)** Yes. According to part (d), the ball is still 20 feet above the ground when it is 382 feet from home plate.

**27.** **(a)** (Assuming that "$x$" is zero at the point of impact.)
$\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where
$x(t) = (35 \cos 27°)t$ and
$y(t) = 4 + (35 \sin 27°)t - 16t^2$.

**(b)** $y_{max} = \dfrac{(v_0 \sin \alpha)^2}{2g} + 4 = \dfrac{(35 \sin 27°)^2}{64} + 4 \approx 7.945$ feet, which is reached at $t = \dfrac{v_0 \sin \alpha}{g} = \dfrac{35 \sin 27°}{32}$
$\approx 0.497$ seconds.

**(c)** For the time, solve $y = 4 + (35 \sin 27°)t - 16t^2 = 0$ for $t$, using the quadratic formula:
$t = \dfrac{35 \sin 27° + \sqrt{(-35 \sin 27°)^2 + 256}}{32}$
$\approx 1.201$ seconds.

Then the range is about $x(1.201) = (35 \cos 27°)(1.201)$
$\approx 37.406$ feet.

**(d)** For the time, solve $y = 4 + (35 \sin 27°)t - 16t^2 = 7$ for $t$, using the quadratic formula:
$t = \dfrac{35 \sin 27° \pm \sqrt{(-35 \sin 27°)^2 - 192}}{32} \approx 0.254$ and
0.740 seconds. At those times the ball is about
$x(0.254) = (35 \cos 27°)(0.254) \approx 7.906$ feet and
$x(0.740) = (35 \cos 27°)(0.740) \approx 23.064$ feet from the impact point, or about $37.460 - 7.906 \approx 29.554$ feet and $37.460 - 23.064 \approx 14.396$ feet from the landing spot.

**(e)** Yes. It changes things because the ball won't clear the net ($y_{max} \approx 7.945$ ft).

**28.** **(a)** $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where

$$x(t) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\cos 20°) \text{ and}$$

$$y(t) = 3 + \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\sin 20°) + \left(\frac{32}{0.12^2}\right)(1 - 0.12t - e^{-0.12t})$$

**(b)** Solve graphically: enter $y(t)$ for $Y_1$ (where $X$ stands in for $t$), then use the maximum function to find that at $t \approx 1.484$ seconds the ball reaches a maximum height of about 40.435 feet.

**(c)** Use the zero function to find that $y = 0$ when the ball has traveled for $\approx 3.126$ seconds. The range is about

$$x(3.126) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12(3.126)})(\cos 20°)$$

$\approx 372.323$ feet.

**(d)** Graph $Y_2 = 30$ and use the intersect function to find that $y = 30$ for $t \approx 0.689$ and 2.305 seconds, at which times the ball is about $x(0.689) \approx 94.513$ feet and $x(2.305) \approx 287.628$ feet from home plate.

**(e)** Yes, the batter has hit a home run since a graph in parametric mode shows that the ball is more than 14 feet above the ground when it passes over the fence.

**29.** **(a)** $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where

$$x(t) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08t})(152 \cos 20° - 17.6) \text{ and}$$

$$y(t) = 3 + \left(\frac{152}{0.08}\right)(1 - e^{-0.08t})(\sin 20°) + \left(\frac{32}{0.08^2}\right)(1 - 0.08t - e^{-0.08t})$$

**(b)** Solve graphically: enter $y(t)$ for $Y_1$ (where $X$ stands in for $t$), then use the maximum function to find that at $t \approx 1.527$ seconds the ball reaches a maximum height of about 41.893 feet.

**(c)** Use the zero function to find that $y = 0$ when the ball has traveled for $\approx 3.181$ seconds. The range is about

$x(3.181)$

$$= \left(\frac{1}{0.08}\right)(1 - e^{-0.08(3.181)})(152 \cos 20° - 17.6)$$

$\approx 351.734$ feet

**(d)** Graph $Y_2 = 35$ and use the interesect function to find that $y = 35$ for $t \approx 0.877$ and 2.190 seconds, at which times the ball is about $\approx x(0.877) \approx 106.028$ feet and $x(2.190) \approx 251.530$ feet from home plate.

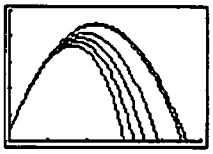
**(e)** No; the range is less than 380 feet. To find the wind needed for a home run, first use the method of part (d) to find that $y = 20$ at $t \approx 0.376$ and 2.716 seconds.

Then define

$$x(w) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08(2.716)})(152 \cos 20° + w),$$

and solve $x(w) = 380$ to find $w \approx 12.846$ ft/sec. This is the speed of a wind gust needed in the direction of the hit for the ball to clear the fence for a home run.

**30.** **(a)** To save time, enter one expression for $x(t)$ in the main window, then use the ENTRY function to repeat it six times in the parametric Y = menu. Do the same for $y(t)$, then make appropriate modifications.



[0, 500] by [0, 50]

Now replace T with X in all the $y(t)$ expressions in the parametric Y = menu, then change to function mode and enter $Y_{1T}$ for $Y_1$, $Y_{2T}$ for $Y_2$, and so on. Use the functions in the CALC menu to fill in the tables for parts (b) and (c).

**(b)**

| drag coeff | time at max ht | max ht |
|---|---|---|
| $k = 0.01$ | $t \approx 1.612$ | 44.777 |
| $k = 0.02$ | $t \approx 1.599$ | 44.336 |
| $k = 0.10$ | $t \approx 1.505$ | 41.149 |
| $k = 0.15$ | $t \approx 1.454$ | 39.419 |
| $k = 0.20$ | $t \approx 1.407$ | 37.854 |
| $k = 0.25$ | $t \approx 1.363$ | 36.431 |

**(c)** After flight times using the zero function, plug the $x$-intercepts into T and read the ranges out as $X_{1T}$, $X_{2T}$, etc.

| drag coeff | flight time | range |
|---|---|---|
| $k = 0.01$ | $t \approx 3.289$ | 462.152 |
| $k = 0.02$ | $t \approx 3.273$ | 452.478 |
| $k = 0.10$ | $t \approx 3.153$ | 386.274 |
| $k = 0.15$ | $t \approx 3.088$ | 352.983 |
| $k = 0.20$ | $t \approx 3.028$ | 324.410 |
| $k = 0.25$ | $t \approx 2.974$ | 299.661 |

**(d)** This follows from the following two limits (as $k \to 0$):

$$\lim_{k\to 0}\frac{1-e^{-kt}}{k} = t, \text{ and}$$
$$\lim_{k\to 0}\frac{1-kt-e^{-kt}}{k^2} = -\frac{t^2}{2}.$$

As $k \to 0$, the air resistance approaches 0.

**31.** The points in question are $(x, y) = \left(\frac{R}{2}, y_{max}\right)$. So,

$x = \frac{v_0^2 \sin\alpha\cos\alpha}{g}$, and $y = \frac{(v_0\sin\alpha)^2}{2g}$. Then

$$x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2$$
$$= \left(\frac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2 + 4\left(\frac{(v_0\sin\alpha)^2}{2g} - \frac{v_0^2}{4g}\right)^2$$
$$= \frac{v_0^4}{g^2}\left[\sin^2\alpha\cos^2\alpha + 4\left(\frac{\sin^2\alpha}{2} - \frac{1}{4}\right)^2\right]$$
$$= \frac{v_0^4}{g^2}\left[\sin^2\alpha\cos^2\alpha + 4\left(\frac{\sin^4\alpha}{4} - \frac{\sin^2\alpha}{4} + \frac{1}{16}\right)\right]$$
$$= \frac{v_0^4}{g^2}\left[\sin^2\alpha\cos^2\alpha + (\sin^2\alpha)(1-\cos^2\alpha) - \sin^2\alpha + \frac{1}{4}\right]$$
$$= \frac{v_0^4}{g^2}\left(\frac{1}{4}\right) = \frac{v_0^4}{4g^2},$$

so the point $(x, y)$ lies on the ellipse.

**32.** $\frac{d\mathbf{r}}{dt} = (v_0e^{-kt}\cos\alpha)\mathbf{i} + (v_0e^{-kt}\sin\alpha + \frac{g}{k}e^{-kt} - \frac{g}{k})\mathbf{j}$

$$\frac{d^2\mathbf{r}}{dt} = (-kv_0e^{-kt}\cos\alpha)\mathbf{i} + (-kv_0e^{-kt}\sin\alpha - ge^{-kt})\mathbf{j}$$
$$= -g\mathbf{j} - k\frac{d\mathbf{r}}{dt}$$

The initial conditions are also satisfied, since

$$\mathbf{r}(0) = \frac{v_0}{k}(1 - e^0)(\cos\alpha)\mathbf{i} + \left[\frac{v_0}{k}(1 - e^0)(\sin\alpha) + \frac{g}{k^2}(1 - 0 - e^0)\right]\mathbf{j}$$
$$= 0\mathbf{i} + 0\mathbf{j} = \mathbf{0},$$

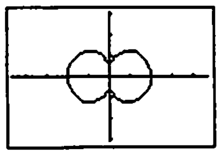
and $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = (v_0e^0\cos\alpha)\mathbf{i} + \left(v_0e^0\sin\alpha + \frac{g}{k}e^0 - \frac{g}{k}\right)\mathbf{j}$

$$= (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j}$$

## ■ Section 10.5 Polar Coordinates and Polar Graphs (pp. 552–559)

### Exploration 1 Investigating Polar Graphs

1. The graph is drawn in the decimal window $[-4.7, 4.7]$ by $[-3.1, 3.1]$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

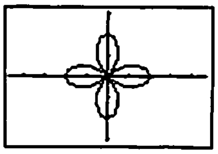


2. $r_1$ and $r_2$ are 0 for $\theta = \frac{\pi}{2}$.
3. $\pi$
4. If $(r, \theta)$ is a solution of $r^2 = 4\cos\theta$, then so is $(r, -\theta)$ because $\cos(-\theta) = \cos\theta$. Thus, the graph is symmetric about the $x$-axis.
If $(r, \theta)$ is a solution of $r^2 = 4\cos\theta$, then so is $(-r, -\theta)$ because $(-r)^2 = r^2$ and $\cos(-\theta) = \cos\theta$. Thus, the graph is symmetric about the $y$-axis.
The graph is symmetric about the origin because it is symmetric about both the $x$- and $y$-axes. You can also give a direct proof by showing that $(-r, \theta)$ lies on the graph if $(r, \theta)$ does.

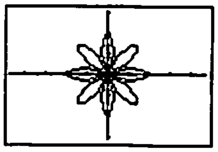
## Exploration 2 Graphing Rose Curves

All graphs are drawn in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.

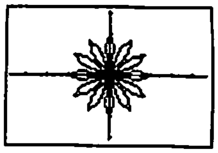
1. The graphs are rose curves with 4 petals when $n = \pm 2$, 8 petals when $n = \pm 4$, and 12 petals when $n = \pm 6$.



$n = \pm 2$

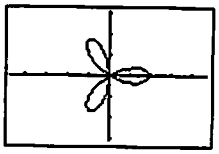


$n = \pm 4$

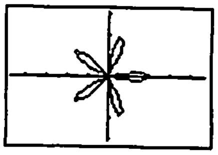


$n = \pm 6$

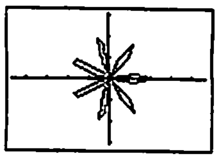
2. $2\pi$
3. The graph is a rose curve with $2|n|$ petals.
4. The graphs are rose curves with 3 petals when $n = \pm 3$, 5 petals when $n = \pm 5$, and 7 petals when $n = \pm 7$.



$n = \pm 3$



$n = \pm 5$



$n = \pm 7$

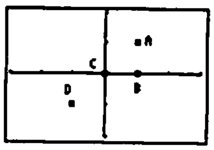
5. $\pi$
6. The graph is a rose cuve with $|n|$ petals.

## Quick Review 10.5

1. Slope $= \dfrac{-1-4}{3-(-2)} = -1$,
   so $y - 4 = -1[x - (-2)]$ or $y = -x + 2$.
2. $(x - 0)^2 + (y - 0)^2 = 3^2$, or $x^2 + y^2 = 9$.
3. $[x - (-2)]^2 + (y - 4)^2 = 2^2$, or $(x + 2)^2 + (y - 4)^2 = 4$.
4. (a) No; $y$ is a function of $x$ and is not the zero function.
   (b) No;
   $y(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) \neq y(x)$
   (c) Yes; $y(-x) = -y(x)$ (see part (b))
5. (a) No; $y$ is a function of $x$ and is not the zero function.
   (b) No; $y(-x) = (-x)^2 - (-x) = x^2 + x \neq y(x)$
   (c) No; $y(-x) \neq -y(x)$ (see part (b))
6. (a) No; $y$ is a function of $x$ and is not the zero function.
   (b) Yes; $y(-x) = \cos(-x) = \cos x = y(x)$
   (c) No; $y(-x) \neq -y(x)$ (see part (b))
7. (a) Yes; Substitute $-y$ for $y$ in the equation to get the original equation.
   (b) Yes; Substitute $-x$ for $x$ in the equation to get the original equation.
   (c) Yes; since the curve is symmetric with respect to both the $x$-axis and $y$-axis, it is symmetric with respect to the origin. (Also, substitute $-x$ for $x$ and $-y$ for $y$ in the equation to get the original equation.)
8. Solve for $y$: $y = (x - 2)^{1/2}$ or $-(x - 2)^{1/2}$.
   Enter the first expression for $Y_1$, the second for $Y_2$.
9. Solve for $y$: $y = \left(\dfrac{4 - x^2}{3}\right)^{1/2}$ or $-\left(\dfrac{4 - x^2}{3}\right)^{1/2}$.
   Enter the first expression for $Y_1$, the second for $Y_2$.
10. $(x^2 - 4x) + (y^2 + 6y + 9) = 0$
    $(x^2 - 4x + 4) + (y^2 + 6y + 9) = 4$
    $(x - 2)^2 + (x + 3)^2 = 2^2$.
    Center $= (2, -3)$, Radius $= 2$.

## Section 10.5 Exercises

For Exercises 1 and 2, two pairs of polar coordinates label the same point if the $r$-coordinates are the same and the $\theta$-coordinates differ by an even multiple of $\pi$, or if the $r$-coordinates are opposites and the $\theta$-coordinates differ by an odd multiple of $\pi$.

1. (a) and (e) are the same.
   (b) and (g) are the same.
   (c) and (h) are the same.
   (d) and (f) are the same.
2. (a) and (f) are the same.
   (b) and (h) are the same.
   (c) and (g) are the same.
   (d) and (e) are the same.
3. 

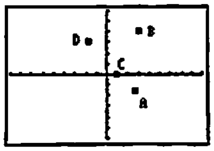
   $[-3, 3]$ by $[-2, 2]$

   (a) $\left(\sqrt{2}\cos\frac{\pi}{4}, \sqrt{2}\sin\frac{\pi}{4}\right) = (1, 1)$

   (b) $(1\cos 0, 1\sin 0) = (1, 0)$

   (c) $\left(0\cos\frac{\pi}{2}, 0\sin\frac{\pi}{2}\right) = (0, 0)$

   (d) $\left(-\sqrt{2}\cos\frac{\pi}{4}, -\sqrt{2}\sin\frac{\pi}{4}\right) = (-1, -1)$

4. 

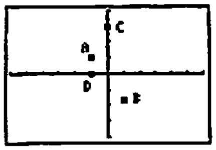
[−9, 9] by [−6, 6]

(a) $\left(-3\cos\frac{5\pi}{6}, -3\sin\frac{5\pi}{6}\right) = \left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

(b) $\left(5\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right), 5\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right) = (3, 4)$

(c) $(-1\cos 7\pi, -1\sin 7\pi) = (1, 0)$

(d) $\left(2\sqrt{3}\cos\frac{2\pi}{3}, 2\sqrt{3}\sin\frac{2\pi}{3}\right) = (-\sqrt{3}, 3)$

5. 

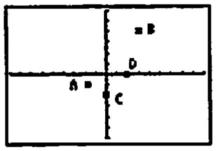
[−6, 6] by [−4, 4]

(a) $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, $\tan\theta = \frac{1}{-1} = -1$ with $\theta$ in quadrant II. The coordinates are $\left(\sqrt{2}, \frac{3\pi}{4}\right)$. $\left(\sqrt{2}, -\frac{5\pi}{4}\right)$ also works, since $r$ is the same and $\theta$ differs by $2\pi$.

(b) $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$, $\tan\theta = -\frac{\sqrt{3}}{1} = -\sqrt{3}$ with $\theta$ in quadrant IV. The coordinates are $\left(2, -\frac{\pi}{3}\right)$. $\left(-2, \frac{2\pi}{3}\right)$ also works, since $r$ has the opposite sign and $\theta$ differs by $\pi$.

(c) $r = \sqrt{0^2 + 3^2} = 3$, $\tan\theta = \frac{3}{0}$ is undefined with $\theta$ on the positive $y$-axis. The coordinates are $\left(3, \frac{\pi}{2}\right)$. $\left(3, \frac{5\pi}{2}\right)$ also works, since $r$ is the same and $\theta$ differs by $2\pi$.

(d) $r = \sqrt{(-1)^2 + 0^2} = 1$, $\tan\theta = \frac{0}{-1} = 0$ with $\theta$ on the negative $x$-axis. The coordinates are $(1, \pi)$. $(-1, 0)$ also works, since $r$ has the opposite sign and $\theta$ differs by $\pi$.

6. 

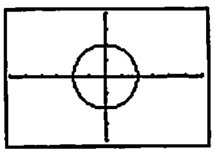
[−9, 9] by [−6, 6]

(a) $r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$, $\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$ with $\theta$ in quadrant III. The coordinates are $\left(2, \frac{7\pi}{6}\right)$. $\left(-2, \frac{\pi}{6}\right)$ also works, since $r$ has the opposite sign and $\theta$ differs by $\pi$.

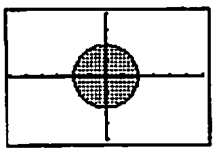
(b) $r = \sqrt{3^2 + 4^2} = 5$, $\tan\theta = \frac{4}{3}$ with $\theta$ in quadrant I. The coordinates are $\left(5, \tan^{-1}\frac{4}{3}\right)$. $\left(-5, \pi + \tan^{-1}\frac{4}{3}\right)$ also works, since $r$ has the opposite sign and $\theta$ differs by $\pi$.

(c) $r = \sqrt{0 + (-2)^2} = 2$, $\tan\theta = -\frac{2}{0}$ is undefined with $\theta$ on the negative $y$-axis. The coordinates are $\left(2, \frac{3\pi}{2}\right)$. $\left(2, -\frac{\pi}{2}\right)$ also works, since $r$ is the same and $\theta$ differs by $2\pi$.

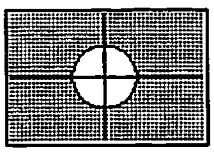
(d) $r = \sqrt{2^2 + 0^2} = 2$, $\tan\theta = \frac{0}{2} = 0$ with $\theta$ on the positive $x$-axis. The coordinates are $(2, 0)$. $(2, 2\pi)$ also works, since $r$ is the same and $\theta$ differs by $2\pi$.

7. 

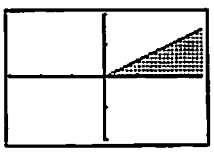
[−6, 6] by [−4, 4]

8. 

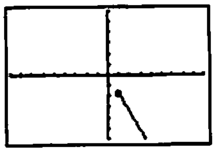
[−6, 6] by [−4, 4]

9. 

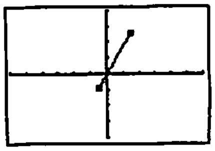
[−3, 3] by [−2, 2]

10. 

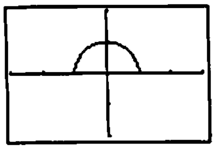
[−3, 3] by [−2, 2]

11. 

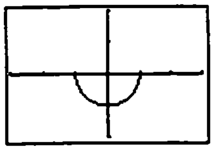
[−9, 9] by [−6, 6]

12. 

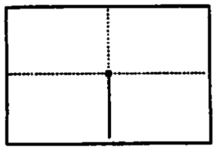
[−6, 6] by [−4, 4]

13. 

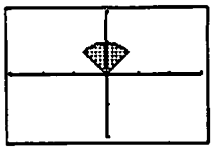
[−3, 3] by [−2, 2]

14. 

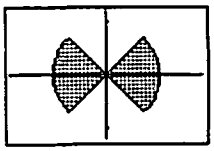
[−3, 3] by [−2, 2]

15. 

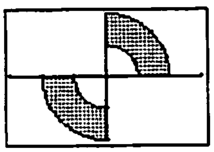
[−3, 3] by [−2, 2]

16. 

[−3, 3] by [−2, 2]

17. 

[−1.8, 1.8] by [−1.2, 1.2]

18. 

[−3, 3] by [−2, 2]

**19.** $y = r\sin\theta$, so the equation is $y = 0$, which is the $x$-axis.

**20.** $x = r\cos\theta$, so the equation is $x = 0$, which is the $y$-axis.

**21.** $r = 4\csc\theta$
$r\sin\theta = 4$
$y = r\sin\theta$, so the equation is $y = 4$, a horizontal line.

**22.** $r = -3\sec\theta$
$r\cos\theta = -3$
$x = r\cos\theta$, so the equation is $x = -3$, a vertical line.

**23.** $x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $x + y = 1$, a line (slope $= -1$, $y$-intercept $= 1$).

**24.** $x^2 + y^2 = r^2$, so the equation is $x^2 + y^2 = 1$, a circle (center $= (0, 0)$, radius $= 1$).

**25.** $x^2 + y^2 = r^2$ and $y = r\sin\theta$, so the equation is $x^2 + y^2 = 4y \Rightarrow x^2 + (y - 2)^2 = 4$, a circle (center $= (0, 2)$, radius $= 2$).

**26.** $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

$r\sin\theta - 2r\cos\theta = 5$

$x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $y - 2x = 5$, a line (slope $= 2$, $y$-intercept $= 5$).

**27.** $r^2\sin 2\theta = 2$

$2r^2\sin\theta\cos\theta = 2$

$(r\sin\theta)(r\cos\theta) = 1$

$x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $xy = 1$ $\left(\text{or, } y = \dfrac{1}{x}\right)$, a hyperbola.

**28.** $r = \cot\theta\csc\theta$

$r\sin\theta = \cot\theta$

$y = r\sin\theta$ and $\dfrac{x}{y} = \cot\theta$, so the equation is $y^2 = x$, a parabola.

**29.** $r = \csc\theta\, e^{r\cos\theta}$
$r\sin\theta = e^{r\cos\theta}$
$x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $y = e^x$, the exponential curve.

**30.** $\cos^2\theta = \sin^2\theta$
$(r\cos\theta)^2 = (r\sin\theta)^2$
$x = r\cos\theta$ and $y = r\sin\theta$, so the equation is $x^2 = y^2$ or $y = \pm x$, the union of two lines.

**31.** $r\sin\theta = \ln r + \ln\cos\theta$
$r\sin\theta = \ln(r\cos\theta)$
$y = \ln x$, the logarithmic curve.

**32.** $r^2 + 2r^2\cos\theta\sin\theta = 1$
$r^2 + 2(r\cos\theta)(r\sin\theta) = 1$
$x^2 + y^2 + 2xy = 1$
$(x + y)^2 = 1$
$x + y = \pm 1$, the union of two lines.

**33.** $r^2 = -4r\cos\theta$
$x^2 + y^2 = -4x$
$(x + 2)^2 + y^2 = 4$, a circle (center $= (-2, 0)$, radius $= 2$).

**34.** $r = 8\sin\theta$
$r^2 = 8r\sin\theta$
$x^2 + y^2 = 8y$
$x^2 + (y - 4)^2 = 16$, a circle (center $= (0, 4)$, radius $= 4$).

**35.** $r = 2\cos\theta + 2\sin\theta$
$r^2 = 2r\cos\theta + 2r\sin\theta$
$x^2 + y^2 = 2x + 2y$
$(x - 1)^2 + (y - 1)^2 = 2$,
a circle (center $= (1, 1)$, radius $= \sqrt{2}$).

**36.** $r \sin\left(\theta + \frac{\pi}{6}\right) = 2$

$r\left(\sin\theta \cos\frac{\pi}{6} + \cos\theta \sin\frac{\pi}{6}\right) = 2$

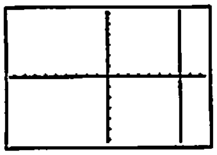
$\frac{\sqrt{3}}{2}r\sin\theta + \frac{1}{2}r\cos\theta = 2$

$\frac{\sqrt{3}}{2}y + \frac{1}{2}x = 2$

$x + \sqrt{3}y = 4$, a line $\left(\text{slope} = -\frac{1}{\sqrt{3}}, \text{ } y\text{-intercept} = \frac{4}{\sqrt{3}}\right)$.

**37.** $x = 7$

$r\cos\theta = 7$. The graph is a vertical line.

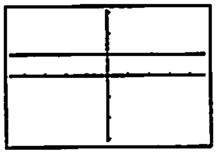


$[-9.4, 9.4]$ by $[-6.2, 6.2]$

**38.** $y = 1$

$r\sin\theta = 1$

The graph is a horiztonal line.

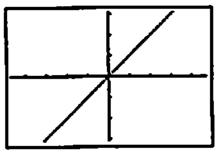


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**39.** $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

More generally, $\theta = \frac{\pi}{4} + 2k\pi$ for any integer $k$.

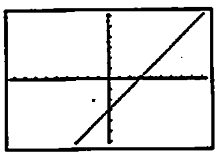
The graph is a slanted line.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**40.** $x - y = 3$

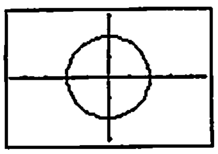
$r\cos\theta - r\sin\theta = 3$



$[-9.4, 9.4]$ by $[-6.2, 6.2]$

**41.** $x^2 + y^2 = 4$

$r^2 = 4$ or $r = 2$ (or $r = -2$)

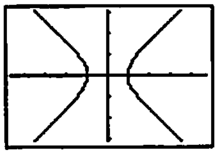


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**42.** $x^2 - y^2 = 1$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$

$r^2(\cos^2\theta - \sin^2\theta) = 1$

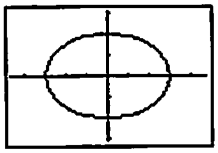


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**43.** $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\frac{r^2\cos^2\theta}{9} + \frac{r^2\sin^2\theta}{4} = 1$

$r^2(4\cos^2\theta + 9\sin^2\theta) = 36$



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

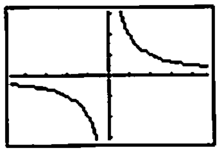
**44.** $xy = 2$

$(r\cos\theta)(r\sin\theta) = 2$

$r^2\cos\theta\sin\theta = 2$

$r^2\, 2\cos\theta\sin\theta = 4$

$r^2\sin 2\theta = 4$

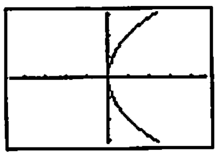


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**45.** $y^2 = 4x$

$r^2\sin^2\theta = 4r\cos\theta$

$r\sin^2\theta = 4\cos\theta$

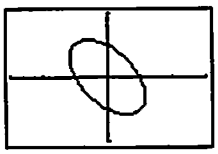


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

**46.** $x^2 + xy + y^2 = 1$

$(r\cos\theta)^2 + (r\cos\theta)(r\sin\theta) + (r\sin\theta)^2 = 1$

$r^2(1 + \cos\theta\sin\theta) = 1$



$[-3, 3]$ by $[-2, 2]$

**47.** $x^2 + (y - 2)^2 = 4$

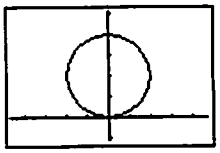
$r^2 \cos^2 \theta + (r \sin \theta - 2)^2 = 4$

$r^2 \cos^2 \theta + r^2 \sin^2 \theta - 4r \sin \theta + 4 = 4$

$r^2 - 4r \sin \theta = 0$

$r = 4 \sin \theta.$

The graph is a circle centered at (0, 2) with radius 2.



[−4.7, 4.7] by [−1.1, 5.1]

**48.** $(x - 3)^2 + (y + 1)^2 = 4$

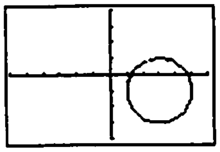
$(r \cos \theta - 3)^2 + (r \sin \theta + 1)^2 = 4$

$r^2 \cos^2 \theta - 6r \cos \theta + 9 + r^2 \sin^2 \theta + 2r \sin \theta + 1 = 4$

$r^2 - 6r \cos \theta + 2r \sin \theta + 6 = 0$

$r = \frac{6 \cos \theta - 2 \sin \theta \pm \sqrt{(6 \cos \theta - 2 \sin \theta)^2 - 24}}{2}$

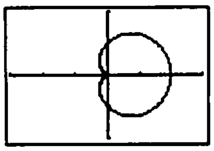
$r = 3 \cos \theta - \sin \theta \pm \sqrt{(3 \cos \theta - \sin \theta)^2 - 6}$



[−6, 6] by [−4, 4]

In Exercises 49–58, find the minimum $\theta$-interval by trying different intervals on a graphing calculator.

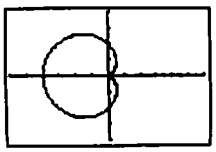
**49. (a)**



[−3, 3] by [−2, 2]

**(b)** Length of interval $= 2\pi$

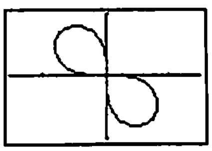
**50. (a)**



[−6, 6] by [−4, 4]

**(b)** Length of interval $= 2\pi$

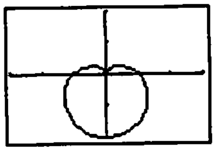
**51. (a)**



[−1.5, 1.5] by [−1, 1]

**(b)** Length of interval $= \frac{\pi}{2}$

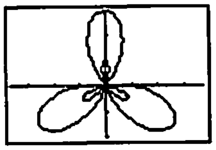
**52. (a)**



[−3, 3] by [−2, 2]

**(b)** Length of interval $= 2\pi$

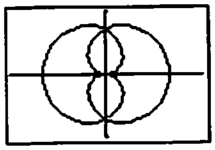
**53. (a)**



[−3.75, 3.75] by [−2, 3]

**(b)** Length of interval $= 2\pi$

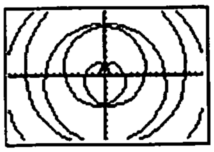
**54. (a)**



[−1.5, 1.5] by [−1, 1]

**(b)** Length of interval $= 4\pi$

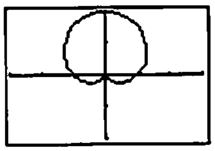
**55. (a)**



[−15, 15] by [−10, 10]

**(b)** Required interval $= (-\infty, \infty)$

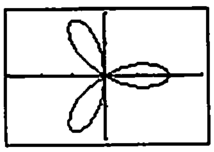
**56. (a)**



[−3, 3] by [−2, 2]

**(b)** Length of interval $= 2\pi$

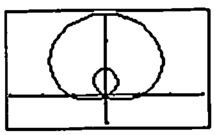
**57. (a)**



[−3, 3] by [−2, 2]

**(b)** Length of interval $= \pi$

**58. (a)**



[−3, 3] by [−1, 3]

**(b)** Length of interval $= 2\pi$

**59.** If $(r, \theta)$ is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. And if $(r, \theta)$ is a solution, so is $(r, -\theta)$. Therefore, the curve is symmetric about the $x$-axis. And since any curve with $x$-axis and origin symmetry also has $y$-axis symmetry, the curve is symmetric about the $y$-axis.

**60.** If $(r, \theta)$ is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. The curve does not have $x$-axis or $y$-axis symmetry.

**61.** If $(r, \theta)$ is a solution, so is $(r, \pi - \theta)$. Therefore, the curve is symmetric about the $y$-axis. The curve does not have $x$-axis or origin symmetry.

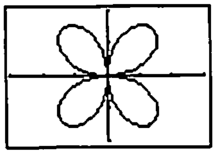
**62.** If $(r, \theta)$ is a solution, so is $(-r, \theta)$. Therefore, the curve is symmetric about the origin. And if $(r, \theta)$ is a solution, so is $(r, -\theta)$. Therefore, the curve is symmetric about the $x$-axis. And since any curve with $x$-axis and origin symmetry also has $y$-axis symmetry, the curve is symmetric about the $y$-axis.

**63.** **(a)** Because $r = a \sec \theta$ is equivalent to $r \cos \theta = a$, which is equivalent to the Cartesian equation $x = a$.

**(b)** $r = a \csc \theta$ is equivalent to $y = a$.

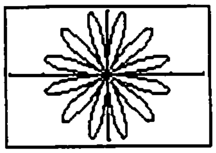
**64.** **(a)** The graph is the same for $n = 2$ and $n = -2$, and in general, it's the same for $n = 2k$ and $n = -2k$. The graphs for $n = 2, 4,$ and $6$ are roses with 4, 8, and 12 "petals" respectively.
The graphs for $n = \pm 2$ and $n = \pm 6$ are shown below.

$n = \pm 2$



$[-3, 3]$ by $[-2, 2]$

$n = \pm 6$

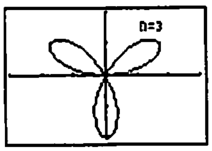


$[-3, 3]$ by $[-2, 2]$

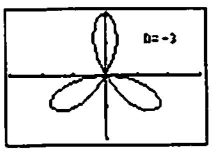
**(b)** $2\pi$

**(c)** The graph is a rose with $2|n|$ "petals".

**(d)** The graphs are roses with 3, 5, and 7 "petals" respectively. The "center petal" points upward if $n = -3, +5,$ or $-7$.
The graphs for $n = 3$ and $n = -3$ are shown below.




$[-3, 3]$ by $[-2, 2]$




$[-3, 3]$ by $[-2, 2]$

**(e)** $\pi$

**(f)** The graph is a rose with $|n|$ "petals".

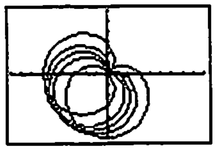
**65.** **(a)** We have $x = r \cos \theta$ and $y = r \sin \theta$. By taking $t = \theta$, we have $r = f(t)$, so $x = f(t) \cos t$ and $y = f(t) \sin t$.

**(b)** $x = 3 \cos t, y = 3 \sin t$

**(c)** $x = (1 - \cos t) \cos t, y = (1 - \cos t) \sin t$

**(d)** $x = (3 \sin 2t) \cos t, y = (3 \sin 2t) \sin t$

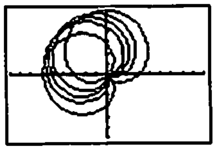
**66.** **(a)**



$[-9, 9]$ by $[-6, 6]$

The graph of $r_2$ is the graph of $r_1$ rotated by angle $\alpha$ counterclockwise about the origin.

**(b)**



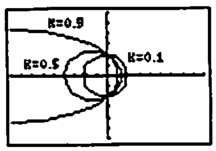
$[-9, 9]$ by $[-6, 6]$

The graph of $r_2$ is the graph of $r_1$ rotated by angle $-\alpha$ clockwise about the origin.

**(c)** The graph of $r_2$ is the graph of $r_1$ rotated counterclockwise about the origin by the angle $\alpha$.

**67.**
$$d = \sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2}$$
$$= [(r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2]^{1/2}$$
$$= [r_2^2 \cos^2 \theta_2 - 2r_2 r_1 \cos \theta_2 \cos \theta_1 + r_1^2 \cos^2 \theta_1$$
$$+ r_2^2 \sin^2 \theta_2 - 2r_2 r_1 \sin \theta_2 \sin \theta_1 + r_1^2 \sin^2 \theta_1]^{1/2}$$
$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_1 - \theta_2)}$$

**68.** **(a)**

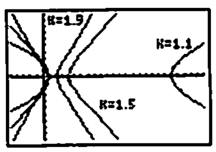



$[-9, 9]$ by $[-6, 6]$

The graphs are ellipses.

**(b)** Graphs for $0 < k < 1$ are ellipses. As $k \to 0^+$, the graph approaches the circle of radius 2 centered at the origin.

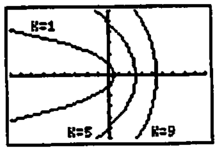
**69.** **(a)**




$[-5, 25]$ by $[-10, 10]$

The graphs are hyperbolas.

**(b)** Graphs for $k > 1$ are hyperbolas. As $k \to 1^+$, the right branch of the hyperbola goes to infinity and "disappears". The left branch approaches the parabola $y^2 = 4 - 4x$.

**70. (a)** 


[−9, 9] by [−6, 6]

The graphs are parabolas.

**(b)** As $k \to 0^+$, the limit of the graph is the negative $x$-axis.

## Section 10.6 Calculus of Polar Curves
(pp. 559–568)

### Quick Review 10.6

**1.** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-3\sin t} = -\frac{5}{3}\cot t$

**2.** $-\frac{5}{3}\cot 2 \approx 0.763$

**3.** Solve $\cot t = 0$: $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$;

the corresponding points are $\left(3\cos\frac{\pi}{2}, 5\sin\frac{\pi}{2}\right) = (0, 5)$

and $\left(3\cos\frac{3\pi}{2}, 5\sin\frac{3\pi}{2}\right) = (0, -5)$

**4.** $-\frac{5}{3}\cot t$ is undefined when $t = 0, \pi,$ or $2\pi$;

the corresponding points are

$(3\cos 0, 5\sin 0) = (3\cos 2\pi, 5\sin 2\pi) = (3, 0)$ and

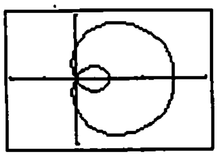
$(3\cos\pi, 5\sin\pi) = (-3, 0)$.

**5.** $\text{Length} = \int_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^{\pi}\sqrt{9\sin^2 t + 25\cos^2 t}\,dt,$

which using NINT evaluates to $\approx 12.763$.

For questions 6–8, the graph is:



[−2, 4] by [−2, 2]

**6.** The upper half of the outer loop

**7.** The inner loop

**8.** The lower half of the outer loop

**9.** $y = 0$ for $x = 0$ or 6.

$\text{Area} = \int_0^6 (6x - x^2)\,dx = \left[3x^2 - \frac{1}{3}x^3\right]_0^6 = 36$

**10.** Use a graphing calculator's intersect function to find that the curves cross at $x \approx 0.270$ and $x \approx 2.248$, then use NINT to find

$\text{Area} = \int_{0.270}^{2.248}[2\sin x - (x^2 - 2x + 1)]\,dx \approx 2.403.$

### Section 10.6 Exercises

**1.** $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

$= \frac{\cos\theta\sin\theta + (-1 + \sin\theta)\cos\theta}{\cos\theta\cos\theta - (-1 + \sin\theta)\sin\theta}$

$= \frac{2\sin\theta\cos\theta - \cos\theta}{\cos^2\theta - \sin^2\theta + \sin\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=0} = -\frac{1}{1} = -1, \left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{1}{1} = 1$

**2.** $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}.$

$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{1}{0}$, which is undefined; $\left.\frac{dy}{dx}\right|_{\theta=\pm\pi/2} = \pm\frac{0}{1} = 0$;

and $\left.\frac{dy}{dx}\right|_{\theta=\pi} = -\frac{1}{0}$, which is undefined.

**3.** $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

$= \frac{-3\cos\theta\sin\theta + (2 - 3\sin\theta)\cos\theta}{-3\cos\theta\cos\theta - (2 - 3\sin\theta)\sin\theta}$

$= \frac{2\cos\theta - 6\sin\theta\cos\theta}{-2\sin\theta - 3(\cos^2\theta - \sin^2\theta)}$

$\left.\frac{dy}{dx}\right|_{(2,0)} = \left.\frac{dy}{dx}\right|_{\theta=0} = \frac{-2}{3} = -\frac{2}{3},$

$\left.\frac{dy}{dx}\right|_{(-1,\pi/2)} = \left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{0}{-1} = 0,$

$\left.\frac{dy}{dx}\right|_{(2,\pi)} = \left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{2}{3}$, and

$\left.\frac{dy}{dx}\right|_{(5,3\pi/2)} = \left.\frac{dy}{dx}\right|_{\theta=3\pi/2} = \frac{0}{-5} = 0.$

**4.** $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

$= \frac{3\sin^2\theta + 3\cos\theta(1 - \cos\theta)}{3\sin\theta\cos\theta - 3\sin\theta(1 - \cos\theta)}$

$= \frac{3\cos\theta - 3(\cos^2\theta - \sin^2\theta)}{6\sin\theta\cos\theta - 3\sin\theta}$

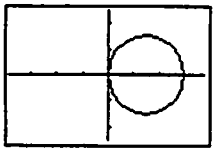
$\left.\frac{dy}{dx}\right|_{(1.5,\pi/3)} = \frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}$, which is undefined;

$\left.\frac{dy}{dx}\right|_{(4.5,2\pi/3)} = \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)}{-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}} = 0;$

$\left.\frac{dy}{dx}\right|_{(6,\pi)} = \frac{-1 - 1}{0 - 0}$, which is undefined; and

$\left.\frac{dy}{dx}\right|_{(3,3\pi/2)} = \frac{0 - (-1)}{0 - (-1)} = 1.$

**5.**



$[-3.8, 3.8]$ by $[-2.5, 2.5]$

The graph passes through the pole when $r = 3\cos\theta = 0$, which occurs when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{3\pi}{2}$. Since the $\theta$-interval $0 \le \pi \le 1$ produces the entire graph, we need only consider $\theta = \frac{\pi}{2}$. At this point, there appears to be a vertical tangent line with equation $\theta = \frac{\pi}{2}$ (or $x = 0$).

Confirm analytically:

$x = (3\cos\theta)\cos\theta = 3\cos^2\theta$

$y = (3\cos\theta)\sin\theta$

$\frac{dy}{d\theta} = (-3\sin\theta)\sin\theta + (3\cos\theta)\cos\theta = 3(\cos^2\theta - \sin^2\theta)$

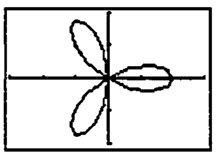
and $\frac{dx}{d\theta} = 6\cos\theta(-\sin\theta)$.

At $\left(0, \frac{\pi}{2}\right)$, $\left.\frac{dx}{d\theta}\right|_{\theta=\pi/2} = 0$, and

$\left.\frac{dy}{d\theta}\right|_{\theta=\pi/2} = 3(0^2 - 1^2) = -3$. So at $\left(0, \frac{\pi}{2}\right)$, $\frac{dx}{d\theta} = 0$

and $\frac{dy}{d\theta} \ne 0$, so $\frac{dy}{dx}$ is undefined and the tangent line is vertical.

**6.**



$[-3, 3]$ by $[-2, 2]$

A trace of the graph suggests three tangent lines, one with positive slope for $\theta = \frac{\pi}{6}$, a vertical one for $\theta = \frac{\pi}{2}$, and one with negative slope for $\theta = \frac{5\pi}{6}$.

Confirm analytically:

$\frac{dy}{d\theta} = -6\sin 3\theta\sin\theta + 2\cos 3\theta\cos\theta$ and

$\frac{dx}{d\theta} = -6\sin 3\theta\cos\theta - 2\cos 3\theta\sin\theta$.

$\left(0, \frac{\pi}{6}\right)$, $\left(0, \frac{\pi}{2}\right)$, and $\left(0, \frac{5\pi}{6}\right)$ are all solutions.

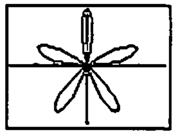
$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, and so

$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-6(1)(1/2) + 2(0)(\sqrt{3}/2)}{-6(1)(\sqrt{3}/2) - 2(0)(1/2)} = \frac{1}{\sqrt{3}}$;

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{-6(-1)(1) + 2(0)(0)}{-6(-1)(0) - 2(0)(1)}$, which is undefined; and

$\left.\frac{dy}{dx}\right|_{\theta=5\pi/6} = \frac{-6(1)(1/2) + 2(0)(-\sqrt{3}/2)}{-6(1)(-\sqrt{3}/2) - 2(0)(1/2)} = -\frac{1}{\sqrt{3}}$. The tangent lines have equations $\theta = \frac{\pi}{6}\left[y = \frac{1}{\sqrt{3}}x\right]$, $\theta = \frac{\pi}{2}\,[x = 0]$, and $\theta = \frac{5\pi}{6}\left[y = -\frac{1}{\sqrt{3}}x\right]$.

**7.**



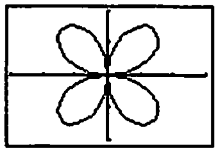
$[-1.5, 1.5]$ by $[-1, 1]$

The polar solutions are $\left(0, \frac{k\pi}{5}\right)$ for $k = 0, 1, 2, 3, 4$, and for a given $k$, the line $\theta = \frac{k\pi}{5}$ appears to be tangent to the curve at $\left(0, \frac{k\pi}{5}\right)$. This can be confirmed analytically by noting that the slope of the curve, $\frac{dy}{dx}$, equals the slope of the line, $\tan\frac{k\pi}{5}$. So the tangent lines are $\theta = 0\,[y = 0]$,

$\theta = \frac{\pi}{5}\left[y = \left(\tan\frac{\pi}{5}\right)x\right]$, $\theta = \frac{2\pi}{5}\left[y = \left(\tan\frac{2\pi}{5}\right)x\right]$,

$\theta = \frac{3\pi}{5}\left[y = \left(\tan\frac{3\pi}{5}\right)x\right]$, and $\theta = \frac{4\pi}{5}\left[y = \left(\tan\frac{4\pi}{5}\right)x\right]$.

**8.**



$[-3, 3]$ by $[-2, 2]$

The polar solutions are $\left(0, \frac{k\pi}{2}\right)$ for $k = 0, 1, 2, 3, 4$, and for a given $k$, the line $\theta = \frac{k\pi}{2}$ appears to be tangent to the curve at $\left(0, \frac{k\pi}{2}\right)$. This can be confirmed analytically by noting that the slope of the curve, $\frac{dy}{dx}$, equals the slope of the line, $\tan\frac{k\pi}{2}$. So the tangent lines are $\theta = 0\,[y = 0]$ and $\theta = \frac{\pi}{2}\,[x = 0]$. $\left(\theta = \pi, \theta = \frac{3\pi}{2}\right.$ and $\theta = 2\pi$ are duplicate solutions.$\left.\right)$

9. $\frac{dy}{d\theta} = \cos\theta\sin\theta + (-1 + \sin\theta)\cos\theta$

$= \cos\theta(2\sin\theta - 1)$

$= \sin 2\theta - \cos\theta$

$\frac{dx}{d\theta} = \cos^2\theta - (-1 + \sin\theta)\sin\theta$

$= \cos^2\theta + \sin\theta - \sin^2\theta$

$= -2\sin^2\theta + \sin\theta + 1$

$\frac{dy}{d\theta} = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ($\cos\theta = 0$) or when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ ($2\sin\theta - 1 = 0$). $\frac{dx}{d\theta} = 0$ when $\sin\theta = \frac{-1 \pm \sqrt{9}}{-4} = -\frac{1}{2}$ or 1, i.e., when $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$, or $\frac{\pi}{2}$. So there is a horizontal tangent line for $\theta = \frac{3\pi}{2}, r = -2$ $\left[\text{the line } y = -2\sin\frac{3\pi}{2} = 2\right]$, for $\theta = \frac{\pi}{6}, r = -\frac{1}{2}$ $\left[\text{the line } y = -\frac{1}{2}\sin\frac{\pi}{6} = -\frac{1}{4}\right]$ and for $\theta = \frac{5\pi}{6}, r = -\frac{1}{2}$ $\left[\text{again, the line } y = -\frac{1}{2}\sin\frac{5\pi}{6} = -\frac{1}{4}\right]$.

There is a vertical tangent line for $\theta = \frac{7\pi}{6}, r = -\frac{3}{2}$ $\left[\text{the line } x = -\frac{3}{2}\cos\frac{7\pi}{6} = \frac{3\sqrt{3}}{4}\right]$ and for $\theta = \frac{11\pi}{6}, r = -\frac{3}{2}$ $\left[\text{the line } x = -\frac{3}{2}\cos\frac{11\pi}{6} = -\frac{3\sqrt{3}}{4}\right]$.

For $\theta = \frac{\pi}{2}, \frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$, but

$\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = 2\cos 2\theta + \sin\theta = -1$ for $\theta = \frac{\pi}{2}$ and

$\frac{d}{d\theta}\left(\frac{dx}{d\theta}\right) = -4\sin\theta\cos\theta + \cos\theta = 0$ for $\theta = \frac{\pi}{2}$, so by L'Hôpital's rule $\frac{dy}{dx}$ is undefined and the tangent line is vertical at $\theta = \frac{\pi}{2}, r = 0$ [the line $x = 0$].

This information can be summarized as follows.

Horizontal at: $\left(-\frac{1}{2}, \frac{\pi}{6}\right)$ $\left[y = -\frac{1}{4}\right]$,

$\left(-\frac{1}{2}, \frac{5\pi}{6}\right)$ $\left[y = -\frac{1}{4}\right]$,

$\left(-2, \frac{3\pi}{2}\right)$ $\left[y = 2\right]$

Vertical at: $\left(0, \frac{\pi}{2}\right)$ [$x = 0$],

$\left(-\frac{3}{2}, \frac{7\pi}{6}\right)$ $\left[x = \frac{3\sqrt{3}}{4}\right]$,

$\left(-\frac{3}{2}, \frac{11\pi}{6}\right)$ $\left[x = -\frac{3\sqrt{3}}{4}\right]$

10. $\frac{dy}{d\theta} = -\sin^2\theta + (1 + \cos\theta)\cos\theta$

$= \cos^2\theta + \cos\theta - \sin^2\theta$

$= 2\cos^2\theta + \cos\theta - 1$

$\frac{dx}{d\theta} = -\sin\theta\cos\theta - (1 + \cos\theta)\sin\theta$

$= -\sin\theta(1 + 2\cos\theta)$

$= -\sin 2\theta - \sin\theta$

$\frac{dy}{d\theta} = 0$ when $\cos\theta = \frac{-1 \pm \sqrt{9}}{4} = -1$ or $\frac{1}{2}$, i.e., when $\theta = \pi, \frac{\pi}{3}$ or $\frac{5\pi}{3}$. $\frac{dx}{d\theta} = 0$ when $\theta = 0, \pi, 2\pi$ (then $\sin\theta = 0$) or when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (then $1 + 2\cos\theta = 0$). So there is a horizontal tangent line for $\theta = \frac{\pi}{3}, r = \frac{3}{2}$ $\left[\text{the line } y = \frac{3}{2}\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{4}\right]$ and for $\theta = \frac{5\pi}{3}, r = \frac{3}{2}$ $\left[\text{the line } y = \frac{3}{2}\sin\frac{5\pi}{3} = -\frac{3\sqrt{3}}{4}\right]$. There is a vertical tangent line for $\theta = 0, r = 2$ [the line $x = 2\cos 0 = 2$], for $\theta = 2\pi, r = 2$ [again, the line $x = 2\cos 2\pi = 2$], for $\theta = \frac{2\pi}{3}, r = \frac{1}{2}$ [the line $x = \frac{1}{2}\cos\frac{2\pi}{3} = -\frac{1}{4}$] and for $\theta = \frac{4\pi}{3}, r = \frac{1}{2}$ $\left[\text{again, the line } x = \frac{1}{2}\cos\frac{2\pi}{3} = -\frac{1}{4}\right]$.

For $\theta = \pi, \frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$, but

$\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = -4\cos\theta\sin\theta - \sin\theta = 0$ for $\theta = \pi$, and

$\frac{d}{d\theta}\left(\frac{dx}{d\theta}\right) = -2\cos 2\theta - \cos\theta = -1$ for $\theta = \pi$,

so by L'Hôpital's rule $\frac{dy}{dx} = 0$ and the tangent line is horizontal at $\theta = \pi, r = 0$ [the line $y = 0$].

This information can be summarized as follows.

Horizontal at: $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ $\left[y = \frac{3\sqrt{3}}{4}\right]$,

$(0, \pi)$ [$y = 0$],

$\left(\frac{3}{2}, \frac{5\pi}{3}\right)$ $\left[y = -\frac{3\sqrt{3}}{4}\right]$

Vertical at: $(2, 0)$ [$x = 2$],

$\left(\frac{1}{2}, \frac{2\pi}{3}\right)$ $\left[x = -\frac{1}{4}\right]$,

$\left(\frac{1}{2}, \frac{4\pi}{3}\right)$ $\left[x = -\frac{1}{4}\right]$,

$(2, 2\pi)$ [$x = 2$]

11. $y = 2\sin^2\theta$

$\frac{dy}{d\theta} = 4\sin\theta\cos\theta$

$= 2\sin 2\theta$

$x = 2\sin\theta\cos\theta$

$= \sin 2\theta$

$\frac{dy}{d\theta} = 2\cos 2\theta$

$\frac{dy}{d\theta} = 0$ when $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{dx}{d\theta} = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. They are never both zero.

For $\theta = 0, \frac{\pi}{2}, \pi$ the curve has horizontal asymptotes at $(0, 0)$ $[y = 0\sin 0 = 0]$, $\left(2, \frac{\pi}{2}\right)\left[y = 2\sin\frac{\pi}{2} = 2\right]$, and $(0, \pi)$ $[y = 0\sin\pi = 0]$. For $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ the curve has vertical asymptotes at $\left(\sqrt{2}, \frac{\pi}{4}\right)$ $[x = \sqrt{2}\cos\frac{\pi}{4} = 1]$ and $\left(\sqrt{2}, \frac{3\pi}{4}\right)\left[x = \sqrt{2}\cos\frac{3\pi}{4} = -1\right]$.

This information can be summarized as follows.

Horizontal at: $(0, 0)$ $[y = 0]$,

$\left(2, \frac{\pi}{2}\right)$ $[y = 2]$,

$(0, \pi)$ $[y = 0]$

Vertical at: $\left(\sqrt{2}, \frac{\pi}{4}\right)$ $[x = 1]$,

$\left(\sqrt{2}, \frac{3\pi}{4}\right)$ $[x = -1]$

12. $\frac{dy}{d\theta} = 4\sin^2\theta + (3 - 4\cos\theta)\cos\theta$

$= 4(\sin^2\theta - \cos^2\theta) + 3\cos\theta$

$= -8\cos^2\theta + 3\cos\theta + 4$

$\frac{dx}{d\theta} = 4\sin\theta\cos\theta - (3 - 4\cos\theta)\sin\theta$

$= \sin\theta(8\cos\theta - 3)$

$= 4\sin 2\theta - 3\sin\theta$

$\frac{dy}{d\theta} = 0$ when $\cos\theta = \frac{-3 \pm \sqrt{137}}{-16}$, i.e., when $\theta \approx 0.405, 2.146, 4.137$, or $5.878$ (values solved for with a graphing calculator). $\frac{dx}{d\theta} = 0$ when $\theta = 0, \pi$ or $2\pi$ (then $\sin\theta = 0$) or when $\theta = \cos^{-1}\left(\frac{3}{8}\right) \approx 1.186$ or $2\pi - \cos^{-1}\left(\frac{3}{8}\right) \approx 5.097$ (then $8\cos\theta - 3 = 0$). So there is a horizontal tangent line for $\theta \approx 0.405$, $r \approx -0.676$ [the line $y \approx -0.676\sin 0.405 \approx -0.267$], for $\theta \approx 2.146$, $r \approx 5.176$ [the line $y \approx 5.176\sin 2.146 \approx 4.343$], for $\theta \approx 4.137$, $r \approx 5.176$ [the line $y \approx 5.176\sin 4.137 \approx -4.343$], and for $\theta \approx 5.878$, $r \approx -0.676$ [the line $y \approx -0.676\sin 5.878 \approx 0.267$]. There is a vertical tangent for $\theta = 0$, $r = -1$ [the line $x = -1\cos 0 = -1$], for $\theta = \pi$, $r = 7$ [the line $x = 7\cos\pi = -7$], for $\theta = 2\pi$, $r = -1$ [again, the line $x = -1\cos 2\pi = -1$], for $\theta = \cos^{-1}\left(\frac{3}{8}\right)$, $r = \frac{3}{2}\left[\text{the line } x = \frac{9}{16}\right]$, and for $\theta = 2\pi - \cos^{-1}\left(\frac{3}{8}\right)$, $r = \frac{3}{2}\left[\text{again, the line } x = \frac{9}{16}\right]$.

This information can be summarized as follows.

Horizontal at: $(-0.676, 0.405)$ $[y \approx -0.267]$,

$(5.176, 2.146)$ $[y \approx 4.343]$,

$(5.176, 4.137)$ $[y \approx -4.343]$,

$(-0.676, 5.878)$ $[y \approx 0.267]$

Vertical at: $(-1, 0)$ $[x = -1]$,

$(1.5, 1.186)$ $\left[x = \frac{9}{16}\right]$,

$(7, \pi)$ $[x = -7]$,

$(1.5, 5.097)$ $\left[x = \frac{9}{16}\right]$,

$(-1, 2\pi)$ $[x = -1]$

13. The curve is complete for $0 \le \theta \le 2\pi$ (as can be verified by graphing). The area is

$$\int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2\, d\theta$$
$$= 2\int_0^{2\pi} (4 + 4\cos\theta + \cos^2\theta)\, d\theta$$
$$= 2\left[4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 18\pi$$

14. The curve is complete for $0 \le \theta \le 2\pi$ (as can be verified by graphing). The area is

$$\int_0^{2\pi} \frac{1}{2}a^2(1 + \cos\theta)^2\, d\theta$$
$$= \frac{1}{2}a^2\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$
$$= \frac{1}{2}a^2\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3}{2}\pi a^2$$

15. Use $r = \sqrt{2a^2\cos 2\theta}$. One lobe is complete for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. The total area is

$$2\int_{-\pi/4}^{\pi/4} \frac{1}{2}(\sqrt{2a^2\cos 2\theta})^2\, d\theta = 2a^2\int_{-\pi/4}^{\pi/4} \cos 2\theta\, d\theta$$
$$= 2a^2\left[\frac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4}$$
$$= 2a^2$$

(Integrating from 0 to $2\pi$ will not work, because $r$ is not defined over the entire interval.)

16. One leaf covers $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. Its area is

$$\int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2 2\theta\, d\theta = \left[\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}.$$

17. Use $r = \sqrt{4\sin 2\theta}$. One loop is complete for $0 \le \theta \le \frac{\pi}{2}$.

Its area is $\int_0^{\pi/2} \frac{1}{2}(\sqrt{4\sin 2\theta})^2\, d\theta = \int_0^{\pi/2} 2\sin 2\theta\, d\theta$

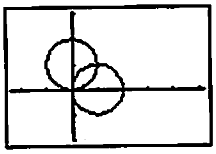
$$= \left[-\cos 2\theta\right]_0^{\pi/2} = 2.$$

18. Use $r = \sqrt{2\sin 3\theta}$. One leaf is complete for $0 \le \theta \le \frac{\pi}{3}$.

The total area is

$$6\int_0^{\pi/3} \frac{1}{2}(\sqrt{2\sin 3\theta})^2\, d\theta = 6\int_0^{\pi/3} \sin 3\theta\, d\theta$$
$$= 2\left[-\cos 3\theta\right]_0^{\pi/3} = 4.$$

19.

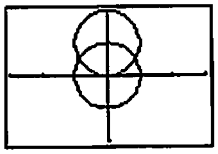


[−2.5, 5.2] by [−2, 3.1]

The circles intersect at $(x, y)$ coordinates (0, 0) and (1, 1). The area shared is twice the area inside the circle $r = 2\sin\theta$ between $\theta = 0$ and $\theta = \frac{\pi}{4}$.

$$\text{Shared area} = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta$$
$$= \int_0^{\pi/4} 4\sin^2\theta\, d\theta$$
$$= 4\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4}$$
$$= 4\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi}{2} - 1.$$

20.

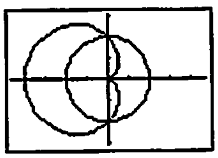


[−3, 3] by [−2, 2]

The circles intersect at $\left(1, \frac{\pi}{6}\right)$ and $\left(1, \frac{5\pi}{6}\right)$. The shared area is $2\int_0^{\pi/6} \frac{1}{2}(2\sin\theta)^2\, d\theta + 2\int_{\pi/6}^{\pi/2} \frac{1}{2}(1)^2\, d\theta$

$$= 4\int_0^{\pi/6} \sin^2\theta\, d\theta + \int_{\pi/6}^{\pi/2} d\theta$$
$$= 4\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6} + \left[\theta\right]_{\pi/6}^{\pi/2} = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi}{3}$$
$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

21.

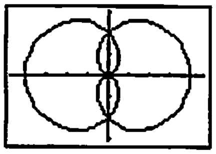


[−4.7, 4.7] by [−3.1, 3.1]

The shared area is half the circle plus two lobelike regions:

$$\frac{1}{2}\pi(2)^2 + 2\int_0^{\pi/2} \frac{1}{2}[2(1 - \cos\theta)]^2\, d\theta$$
$$= 2\pi + \int_0^{\pi/2} (4 - 8\cos\theta + 4\cos^2\theta)\, d\theta$$
$$= 2\pi + 4\left[\theta - 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$
$$= 5\pi - 8$$

22.

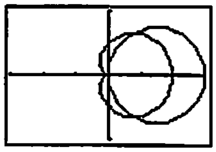


[−4.7, 4.7] by [−3.1, 3.1]

Use the symmetries of the graphs: the shared area is

$$4\int_0^{\pi/2}\frac{1}{2}[2(1-\cos\theta)]^2\,d\theta$$
$$=8\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$$
$$=8\left[\theta-2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi/2}=6\pi-16$$

23. For $a = 1$:



[−3, 3] by [−2, 2]

The curves intersect at the origin and when

$3a\cos\theta = a(1+\cos\theta)$

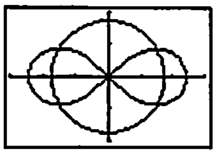
$2\cos\theta = 1$

$\theta = \pm\frac{\pi}{3}$.

Use the symmetries of the curves: the area in question is

$$2\int_0^{\pi/3}\frac{1}{2}[(3a\cos\theta)^2-a^2(1+\cos\theta)^2]\,d\theta$$
$$=a^2\int_0^{\pi/3}(9\cos^2\theta-1-2\cos\theta-\cos^2\theta)\,d\theta$$
$$=a^2\int_0^{\pi/3}(8\cos^2\theta-2\cos\theta-1)\,d\theta$$
$$=a^2\left[4\theta+2\sin 2\theta-2\sin\theta-\theta\right]_0^{\pi/3}=a^2\pi$$

24.



[−3, 3] by [−2, 2]

The curves intersect when

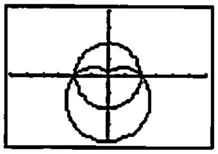
$6\cos 2\theta = 3$

$\theta = \pm\frac{\pi}{6}$ or $\pm\frac{5\pi}{6}$.

Use the symmetries of the curves. The area in question is

$$4\int_0^{\pi/6}\frac{1}{2}(6\cos 2\theta-3)\,d\theta=6\left[\sin 2\theta-\theta\right]_0^{\pi/6}=3\sqrt{3}-\pi.$$

25.

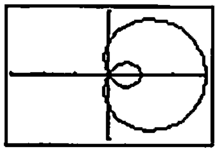


[−6, 6] by [−4, 4]

The area in question is half the circle minus two lobelike regions:

$$\frac{1}{2}\pi(2)^2-2\int_0^{\pi/2}\frac{1}{2}[2(1-\sin\theta)]^2\,d\theta$$
$$=2\pi-\int_0^{\pi/2}(4-8\sin\theta+4\sin^2\theta)\,d\theta$$
$$=2\pi-4\left[\theta+2\cos\theta+\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{\pi/2}=8-\pi$$

26.



[−3, 3] by [−2, 2]

(a) To find the integration limits, solve

$2\cos\theta + 1 = 0$

$\theta = \pm\frac{2\pi}{3}$.

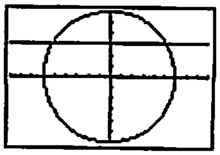
Because of the curve's symmetry, the area inside the outer loop is

$$2\int_0^{2\pi/3}\frac{1}{2}(2\cos\theta+1)^2\,d\theta$$
$$=\int_0^{2\pi/3}(4\cos^2\theta+4\cos\theta+1)\,d\theta$$
$$=\left[2\theta+\sin 2\theta+4\sin\theta+\theta\right]_0^{2\pi/3}$$
$$=\frac{3\sqrt{3}}{2}+2\pi$$

(b) Again, use the curve's symmetry. The inner loop's area is

$$2\int_{2\pi/3}^{\pi}\frac{1}{2}(2\cos\theta+1)^2\,d\theta$$
$$=\left[2\theta+\sin 2\theta+4\sin\theta+\theta\right]_{2\pi/3}^{\pi}=\pi-\frac{3\sqrt{3}}{2}.$$

Subtract this from the answer in (a) to get $3\sqrt{3}+\pi$.

27. 

$[-9, 9]$ by $[-6, 6]$

To find the integration limits, solve

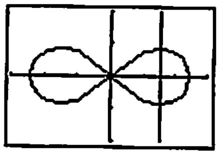
$3 \csc \theta = 6$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. The area in question is

$$\int_{\pi/6}^{5\pi/6} \frac{1}{2}(6^2 - 3^2 \csc^2 \theta)\, d\theta$$

$$= \frac{1}{2}\Big[36\theta + 9 \cot \theta\Big]_{\pi/6}^{5\pi/6}$$

$$= 12\pi - 9\sqrt{3}$$

28. 

$[-3, 3]$ by $[-2, 2]$

To find the intersection points, solve

$6 \cos 2\theta = \frac{9}{4} \sec^2 \theta$

$48 \cos^4 \theta - 24 \cos^2 \theta - 9 = 0$

$\cos^2 \theta = \frac{3}{4}$

$\theta = \pm\frac{\pi}{6}$.

By the symmetry of the curves, the area in question is

$$2\int_0^{\pi/6} \frac{1}{2}\left(6 \cos 2\theta - \frac{9}{4} \sec^2 \theta\right) d\theta$$

$$= \left[3 \sin 2\theta - \frac{9}{4} \tan \theta\right]_0^{\pi/6} = \frac{3\sqrt{3}}{4}.$$

29. (a) Find the area of the right half in two parts, then double the result: Right half area

$$= \int_0^{\pi/4} \frac{1}{2} \tan^2 \theta\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\left(\frac{1}{2} \csc^2 \theta\right) d\theta$$

$$= \frac{1}{2}\Big[\tan \theta - \theta\Big]_0^{\pi/4} + \frac{1}{4}\Big[-\cot \theta\Big]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\left(1 - \frac{\pi}{4}\right) + \frac{1}{4}(0 + 1) = \frac{3}{4} - \frac{\pi}{8}.$$

Total area is twice that, or $\frac{3}{2} - \frac{\pi}{4}$.

(b) Yes. $x = \tan \theta \cos \theta \Rightarrow x = \sin \theta$

$y = \tan \theta \sin \theta \Rightarrow y = \frac{\sin^2 \theta}{\cos \theta}$

$\lim_{\theta \to -\pi/2^+} x = -1, \quad \lim_{\theta \to -\pi/2^+} y = \infty$

$\lim_{\theta \to \pi/2^-} x = 1, \quad \lim_{\theta \to \pi/2^-} y = \infty$

30. The integral given is incorrect because $r = \cos \theta$ sweeps out the circle twice as $\theta$ goes from 0 to $2\pi$. Or, you can't use equation (2) from the text on the interval $[0, 2\pi]$ because $r = \cos \theta$ is negative for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. The correct area is $\frac{5\pi}{4}$, which can be found by computing the areas of the cardioid $\frac{3\pi}{2}$ and the circle $\frac{\pi}{4}$ separately and subtracting.

31. $\frac{dr}{d\theta} = 2\theta$, so

$$\text{Length} = \int_0^{\sqrt{5}} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta$$

$$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta$$

$$= \left[\frac{1}{3}(\theta^2 + 4)^{3/2}\right]_0^{\sqrt{5}}$$

$$= \frac{1}{3}(27 - 8) = \frac{19}{3}$$

32. $\frac{dr}{d\theta} = \frac{e^\theta}{\sqrt{2}}$, so

$$\text{Length} = \int_0^{\pi} \sqrt{\left(\frac{e^\theta}{\sqrt{2}}\right)^2 + \left(\frac{e^\theta}{\sqrt{2}}\right)^2}\, d\theta$$

$$= \int_0^{\pi} e^\theta\, d\theta$$

$$= \Big[e^\theta\Big]_0^{\pi} = e^\pi - 1$$

33. $\frac{dr}{d\theta} = -\sin \theta$, so

$$\text{Length} = \int_0^{2\pi} \sqrt{(1 + \cos \theta)^2 + (-\sin \theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2 \cos \theta}\, d\theta,$$

$$= \int_0^{2\pi} \sqrt{2 + 4 \cos^2\left(\frac{\theta}{2}\right) - 2}\, d\theta$$

$$= \int_0^{2\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

$$= 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 8\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 8$$

34. $\frac{dr}{d\theta} = a \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)$, so

$$\text{Length} = \int_0^{\pi} \sqrt{a^2 \sin^4\left(\frac{\theta}{2}\right) + a^2 \sin^2\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= a\int_0^{\pi} \sqrt{\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 2a\left[-\cos\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 2a$$

35. $\frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 + \cos \theta)^2}$, so

Length $= \int_0^{\pi/2} \sqrt{\frac{6^2}{(1 + \cos \theta)^2} + \frac{6^2 \sin^2 \theta}{(1 + \cos \theta)^4}}\, d\theta$, which using

NINT evaluates to $\approx 6.887$.

(Note: the integrand can simplify to $3 \sec^3\left(\frac{\theta}{2}\right)$.)

36. $\frac{dr}{d\theta} = -\frac{2 \sin \theta}{(1 - \cos \theta)^2}$, so

Length $= \int_{\pi/2}^{\pi} \sqrt{\frac{2^2}{(1 - \cos \theta)^2} + \frac{2^2 \sin^2 \theta}{(1 - \cos \theta)^4}}\, d\theta$, which using

NINT evalutes to $\approx 2.296$.

(Note: the integrand can simplify to $\csc^3\left(\frac{\theta}{2}\right)$).

37. $\frac{dr}{d\theta} = -\cos^2\left(\frac{\theta}{3}\right) \sin\left(\frac{\theta}{3}\right)$, so

$$\text{Length} = \int_0^{\pi/4} \sqrt{\cos^6\left(\frac{\theta}{3}\right) + \cos^4\left(\frac{\theta}{3}\right)\sin^2\left(\frac{\theta}{3}\right)}\, d\theta$$

$$= \int_0^{\pi/4} \sqrt{\cos^4\left(\frac{\theta}{3}\right)}\, d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{3}{4}\sin\left(\frac{2\theta}{3}\right)\right]_0^{\pi/4} = \frac{\pi + 3}{8}.$$

38. $\frac{dr}{d\theta} = \frac{\cos 2\theta}{\sqrt{1 + \sin 2\theta}}$, so

$$\text{Length} = \int_0^{\pi\sqrt{2}} \sqrt{\frac{\cos^2 2\theta}{1 + \sin 2\theta} + 1 + \sin 2\theta}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{\frac{\cos^2 2\theta + (1 + \sin 2\theta)^2}{1 + \sin 2\theta}}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{2}\, d\theta = 2\pi.$$

39. $\frac{dr}{d\theta} = \frac{1}{2\sqrt{\cos 2\theta}}(-\sin 2\theta)(2) = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$, so

Surface area

$$= \int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \cos \theta \sqrt{(\sqrt{\cos 2\theta})^2 + \left(-\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2}\, d\theta$$

$$= 2\pi \int_0^{\pi/4} \cos \theta \sqrt{\cos^2 2\theta + \sin^2 2\theta}\, d\theta$$

$$= 2\pi \int_0^{\pi/4} \cos \theta\, d\theta$$

$$= 2\pi\Big[\sin \theta\Big]_0^{\pi/4} = \pi\sqrt{2} \approx 4.443.$$

40. $\frac{dr}{d\theta} = \left(\frac{\sqrt{2}}{2}\right)e^{\theta/2}$, so surface area

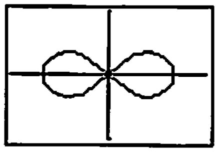
$$= \int_0^{\pi/2} 2\pi\sqrt{2}e^{\theta/2} \sin \theta \sqrt{(\sqrt{2}e^{\theta/2})^2 + \left(\frac{\sqrt{2}}{2}\right)^2 (e^{\theta/2})^2}\, d\theta$$

$$= \int_0^{\pi/2} 2\pi e^{\theta} \sin \theta \sqrt{5}\, d\theta$$

$$= 2\sqrt{5}\pi\left[\frac{1}{2}e^{\theta}(\sin \theta - \cos \theta)\right]_0^{\pi/2}$$

$$= \sqrt{5}\pi(e^{\pi/2} + 1) \approx 40.818$$

41.



[−1.5, 1.5] by [−1, 1]

$$2r\frac{dr}{d\theta} = -2 \sin 2\theta$$

$$\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}.$$

Use the curve's symmetry and note that $r$ is defined for

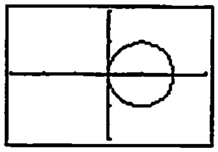
$0 \le \theta \le \frac{\pi}{4}$: Surface area

$$= 2\int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \sin \theta \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 2\int_0^{\pi/4} 2\pi \sin \theta\, d\theta$$

$$= 4\pi\Big[-\cos \theta\Big]_0^{\pi/4} = (4 - 2\sqrt{2})\pi \approx 3.681$$

42. For $a = 1$:



[−3, 3] by [−2, 2]

$\frac{dr}{d\theta} = -2a \sin \theta$, so surface area

$$= 2\int_0^{\pi/2} 2\pi(2a \cos \theta) \cos \theta\sqrt{4a^2 \cos^2 \theta + 4a^2 \sin^2 \theta}\, d\theta$$

$$= 16a^2\pi\int_0^{\pi/2} \cos^2 \theta\, d\theta$$

$$= 16a^2\pi\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = 4a^2\pi^2$$

43. $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$

$$= (f'(\theta) \cos \theta - f(\theta) \sin \theta)^2$$

$$\quad + (f'(\theta) \sin \theta + f(\theta) \cos \theta)^2$$

$$= (f'(\theta) \cos \theta)^2 + (f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta)^2$$

$$\quad + (f(\theta) \cos \theta)^2$$

$$= (f(\theta))^2(\cos^2 \theta + \sin^2 \theta)$$

$$\quad + (f'(\theta))^2(\cos^2 \theta + \sin^2 \theta)$$

$$= (f(\theta))^2 + (f'(\theta))^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

44. (a) $\frac{1}{2\pi - 0}\int_0^{2\pi} a(1 - \cos \theta)\, d\theta = \frac{a}{2\pi}\Big[\theta - \sin \theta\Big]_0^{2\pi} = a$

(b) $\frac{1}{2\pi - 0}\int_0^{2\pi} a\, d\theta = \frac{a}{2\pi}\Big[\theta\Big]_0^{2\pi} = a$

(c) $\frac{1}{\pi/2 - (-\pi/2)}\int_{-\pi/2}^{\pi/2} a \cos \theta\, d\theta = \frac{a}{\pi}\Big[\sin \theta\Big]_{-\pi/2}^{\pi/2} = \frac{2a}{\pi}$

45. If $g(\theta) = 2f(\theta)$, then

$\sqrt{(g(\theta)^2 + g'(\theta)^2)} = 2\sqrt{(f(\theta)^2 + f'(\theta)^2)}$, so the length of $g$ is 2 times the length of $f$.

**46.** If $g(\theta) = 2f(\theta)$, then
$2\pi\, g(\theta) \sin\theta \sqrt{(g(\theta))^2 + g'(\theta)^2)}$
$= 4[2\pi f(\theta) \sin\theta \sqrt{(f(\theta))^2 + f'(\theta)^2)}]$,
so the area generated by $g$ is 4 times that of $f$.

**47.** **(a)** Let $r = 1.75 + \dfrac{0.06\theta}{2\pi}$.

**(b)** Since $\dfrac{dr}{d\theta} = \dfrac{b}{2\pi}$, this is just Equation 4 for the length of the curve.

**(c)** Using NINT, $\displaystyle\int_0^{80\pi} \sqrt{\left(1.75 + \frac{0.06\theta}{2\pi}\right)^2 + \left(\frac{0.06}{2\pi}\right)^2}\, d\theta$ evaluates to $\approx 741.420$ cm $\approx 7.414$ m.

**(d)** $\left(r^2 + \left(\dfrac{b}{2\pi}\right)^2\right)^{1/2} = r\left(1 + \left(\dfrac{b}{2\pi r}\right)^2\right)^{1/2} \approx r$
since $\left(\dfrac{b}{2\pi r}\right)^2$ is a very small quantity squared.

**(e)** $L \approx 741.420$ cm (from part (c)),

$$L_a = \int_0^{80\pi} \left(1.75 + \frac{0.06\theta}{2\pi}\right) d\theta$$
$$= \left[1.75\theta + \frac{0.03\theta^2}{2\pi}\right]_0^{80\pi} = 236\pi \approx 741.416 \text{ cm}$$

**48.** **(a)** Use the approximation, $L_a$, from 47(e). If the reel has made $n$ complete turns, then the angle is $2\pi n$. So from the integral, $L_a = \pi b n^2 + 2\pi r_0 n$. Solving for $n$ gives
$$n = \left(\frac{r_0}{b}\right)\left(\sqrt{\frac{bL}{r_0^2\pi} + 1} - 1\right).$$

**(b)** The take up reel slows down as time progresses.

**(c)** Since $L$ is proportional to time, the formula in part (a) shows that $n$ will grow roughly as the square root of time.

**49.** $$\frac{2}{3}\int_0^{2\pi} a^3(1 + \cos\theta)^3 \cos\theta\, d\theta$$
$$= \frac{2}{3}a^3 \int_0^{2\pi} (\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\, d\theta$$
$$= \frac{2}{3}a^3\left[3\sin\theta + \frac{15}{8}\sin\theta\cos\theta + \frac{15}{8}\theta + \cos^2\theta\sin\theta + \frac{1}{4}\cos^3\theta\sin\theta\right]_0^{2\pi}$$
$= \dfrac{5}{2}\pi a^3$, and $\displaystyle\int_0^{2\pi} a^2(1 + \cos\theta)^2\, d\theta$
$$= a^2\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$
$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi a^2.$$
So $\bar{x} = \dfrac{\frac{5}{2}\pi a^3}{3\pi a^2} = \dfrac{5a}{6}$.

By symmetry, $\bar{y} = 0$, so the centroid is $\left(\dfrac{5a}{6}, 0\right)$.

**50.** $\dfrac{2}{3}\displaystyle\int_0^{\pi} a^3 \sin\theta\, d\theta = \frac{2}{3}a^3\Big[-\cos\theta\Big]_0^{\pi} = \frac{4}{3}a^3$, and
$\displaystyle\int_0^{\pi} a^2\, d\theta = \pi a^2$. So
$\bar{y} = \dfrac{\frac{4}{3}a^3}{\pi a^2} = \dfrac{4a}{3\pi}$. By symmetry, $\bar{x} = 0$, so the centroid is $\left(0, \dfrac{4a}{3\pi}\right)$.

## ■ Chapter 10 Review Exercises
(pp. 569–572)

**1.** **(a)** $3\langle -3, 4\rangle - 4\langle 2, -5\rangle = \langle -9 - 8, 12 + 20\rangle$
$= \langle -17, 32\rangle$

**(b)** $\sqrt{17^2 + 32^2} = \sqrt{1313}$

**2.** **(a)** $\langle -3 + 2, 4 - 5\rangle = \langle -1, -1\rangle$

**(b)** $\sqrt{1^2 + 1^2} = \sqrt{2}$

**3.** **(a)** $\langle -2(-3), -2(4)\rangle = \langle 6, -8\rangle$

**(b)** $\sqrt{6^2 + 8^2} = 10$

**4.** **(a)** $\langle 5(2), 5(-5)\rangle = \langle 10, -25\rangle$

**(b)** $\sqrt{10^2 + 25^2} = \sqrt{725} = 5\sqrt{29}$

**5.** $\dfrac{\pi}{6}$ radians below the negative $x$-axis: $\left\langle -\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right\rangle$ [assuming counterclockwise].

**6.** $\left\langle \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right\rangle$

**7.** $2\left(\dfrac{1}{\sqrt{4^2 + 1^2}}\right)\langle 4, -1\rangle = \left\langle \dfrac{8}{\sqrt{17}}, -\dfrac{2}{\sqrt{17}}\right\rangle$

**8.** $-5\left(\dfrac{1}{\sqrt{(3/5)^2 + (4/5)^2}}\right)\left\langle \dfrac{3}{5}, \dfrac{4}{5}\right\rangle = \langle -3, -4\rangle$

**9.** **(a)** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{(1/2)\sec t \tan t}{(1/2)\sec^2 t} = \sin t$

For $t = \dfrac{\pi}{3}$; $x = \dfrac{\sqrt{3}}{2}$, $y = 1$, and $\dfrac{dy}{dx} = \dfrac{\sqrt{3}}{2}$. So the tangent line is $y - 1 = \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\sqrt{3}}{2}\right)$ or
$y = \dfrac{\sqrt{3}}{2}x + \dfrac{1}{4}$.

**(b)** $\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dx} = \dfrac{dy'/dt}{dx/dt} = \dfrac{\cos t}{(1/2)\sec^2 t} = 2\cos^3 t$, which for $t = \dfrac{\pi}{3}$ equals $\dfrac{1}{4}$.

**10.** **(a)** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3/t^2}{-2/t^3} = -\dfrac{3}{2}t$

For $t = 2$: $x = \dfrac{5}{4}$, $y = -\dfrac{1}{2}$, and $\dfrac{dy}{dx} = -3$.

So the tangent line is $y + \dfrac{1}{2} = -3\left(x - \dfrac{5}{4}\right)$ or
$y = -3x + \dfrac{13}{4}$.

**(b)** $\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dx} = \dfrac{dy'/dt}{dx/dt} = \dfrac{-3/2}{-2/t^3} = \dfrac{3t^3}{4}$, which for $t = 2$ equals 6.

11. $\frac{dy}{dt} = \frac{1}{2}\tan t \sec t$ equals zero for $t = k\pi$, where $k$ is any integer. $\frac{dx}{dt} = \frac{1}{2}\sec^2 t$ never equals zero.

(a) Horizontal tangents at $\left(\frac{1}{2}\tan 0, \frac{1}{2}\sec 0\right) = \left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}\tan \pi, \frac{1}{2}\sec \pi\right) = \left(0, -\frac{1}{2}\right)$.

(b) There are no vertical tangents, since $\frac{dx}{dt}$ never equals zero.

12. $\frac{dy}{dt} = 2\cos t$ equals zero for $\theta = \frac{k\pi}{2}$, where $k$ is any odd integer. $\frac{dx}{dt} = 2\sin t$ equals zero for $t = k\pi$, where $k$ is any integer.

(a) Horizontal tangents at $\left(-2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}\right) = (0, 2)$ and $\left(-2\cos\frac{3\pi}{2}, 2\sin\frac{3\pi}{2}\right) = (0, -2)$.

(b) Vertical tangents at $(-2\cos 0, 2\sin 0) = (-2, 0)$ and $(-2\cos\pi, 2\sin\pi) = (2, 0)$.

13. $\frac{dy}{dt} = -2\sin t\cos t = -\sin 2t$ equals zero for $t = \frac{k\pi}{2}$, where $k$ is any integer. $\frac{dx}{dt} = \sin t$ equals zero for $t = k\pi$, where $k$ is any integer. Where they are both zero, use L'Hôpital's rule:

$$\lim_{t\to k\pi}\frac{dy/dt}{dx/dt} = \lim_{t\to k\pi}\frac{-\sin 2t}{\sin t} = \lim_{t\to k\pi}\frac{-2\cos 2t}{\cos t} = \pm 2.$$

(a) Horizontal tangent at $\left(-\cos\frac{\pi}{2}, \cos^2\frac{\pi}{2}\right) = (0, 0)$.

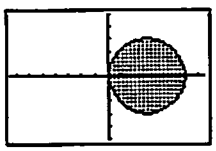
(b) There are no vertical tangents.

14. $\frac{dy}{dt} = 9\cos t$ equals zero for $t = \frac{k\pi}{2}$, where $k$ is any odd integer. $\frac{dx}{dt} = -4\sin t$ equals zero for $t = k\pi$, where $k$ is any integer.

(a) Horizontal tangents at $\left(4\cos\frac{\pi}{2}, 9\sin\frac{\pi}{2}\right) = (0, 9)$ and $\left(4\cos\frac{3\pi}{2}, 9\sin\frac{3\pi}{2}\right) = (0, -9)$.

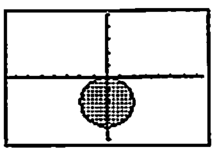
(b) Vertical tangents at $(4\cos 0, 9\sin 0) = (4, 0)$ and $(4\cos\pi, 9\sin\pi) = (-4, 0)$.

15.



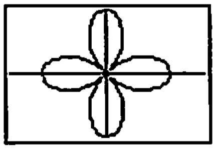
[−7.5, 7.5] by [−5, 5]

16.



[−7.5, 7.5] by [−5, 5]

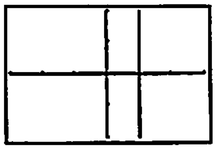
17. (a)



[−1.5, 1.5] by [−1, 1]

(b) $2\pi$

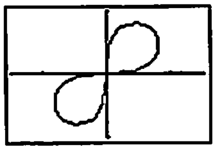
18. (a)



[−3, 3] by [−2, 2]

(b) $\pi$

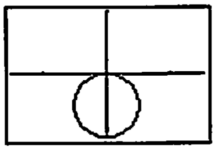
19. (a)



[−1.5, 1.5] by [−1, 1]

(b) $\frac{\pi}{2}$

20. (a)



[−1.5, 1.5] by [−1, 1]

(b) $\pi$

21. $$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{-2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}$$

$\left(0, \frac{\pi}{4}\right), \left(0, \frac{3\pi}{4}\right), \left(0, \frac{5\pi}{4}\right)$ and $\left(0, \frac{7\pi}{4}\right)$ are polar solutions.

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-2/\sqrt{2}}{-2/\sqrt{2}} = 1, \left.\frac{dy}{dx}\right|_{\theta=3\pi/4} = \frac{2/\sqrt{2}}{-2/\sqrt{2}} = -1,$$

$$\left.\frac{dy}{dx}\right|_{\theta=5\pi/4} = \frac{2/\sqrt{2}}{2/\sqrt{2}} = 1, \left.\frac{dy}{dx}\right|_{\theta=7\pi/4} = \frac{-2\sqrt{2}}{2\sqrt{2}} = -1.$$

The Cartesian equations are $y = \pm x$.

22. $$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \frac{-2\sin 2\theta\sin\theta + (1 + \cos 2\theta)\cos\theta}{-2\sin 2\theta\cos\theta - (1 + \cos 2\theta)\sin\theta}$$

$$= \frac{-4\sin^2\theta\cos\theta + \cos\theta + 2\cos^3\theta - \cos\theta}{-4\cos^2\theta\sin\theta - \sin\theta - 2\cos^2\theta\sin\theta + \sin\theta}$$

$$= \frac{-4\sin^2\theta + 2\cos^2\theta}{-6\cos\theta\sin\theta}$$

$$= \frac{4\sin^2\theta - 2\cos^2\theta}{3\sin 2\theta}.$$

$\left(0, \frac{\pi}{2}\right)$ and $\left(0, \frac{3\pi}{2}\right)$ are polar solutions.

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy}{dx}\right|_{\theta=3\pi/2} = \frac{4}{0}$ is undefined, so the tangent lines are vertical with equation $x = 0$.

**23.** $\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\left(1 - \cos\left(\frac{\theta}{2}\right)\right)\sin\theta\right]$

$= \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\sin\theta + \cos\theta - \cos\left(\frac{\theta}{2}\right)\cos\theta$

$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[\left(1 - \cos\left(\frac{\theta}{2}\right)\right)\cos\theta\right]$

$= \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos\theta - \sin\theta + \cos\left(\frac{\theta}{2}\right)\sin\theta$

Solve $\frac{dy}{d\theta} = 0$ for $\theta$ with a graphing calculator: the solutions are 0, $\approx 2.243$, $\approx 4.892$, $\approx 7.675$, $\approx 10.323$, and $4\pi$. Using the middle four solutions to $y = r\sin\theta$ reveals horizontal tangent lines at $y \approx \pm 0.443$ and $y \approx \pm 1.739$.

Solve $\frac{dx}{d\theta} = 0$ for $\theta$ with a graphing calculator: the solutions are 0, $\approx 1.070$, $\approx 3.531$, $2\pi$, $\approx 9.035$, $\approx 11.497$, and $4\pi$.

Using the middle five solutions to find $x = r\cos\theta$ reveals vertical tangent lines at $x = 2$, $x \approx 0.067$, and $x \approx -1.104$.

Where $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both equal zero ($\theta = 0, 4\pi$), close inspection of the plot shows that the tangent lines are horizontal, with equation $y = 0$. (This can be confirmed using L'Hôpital's rule.)

**24.** $\frac{dy}{d\theta} = \frac{d}{d\theta}[2(1 - \sin\theta)\sin\theta] = -4\sin\theta\cos\theta + 2\cos\theta$

$\frac{dx}{d\theta} = \frac{d}{d\theta}[2(1 - \sin\theta)\cos\theta]$

$= -2\cos^2\theta - 2\sin\theta + 2\sin^2\theta$

$= 4\sin^2\theta - 2\sin\theta - 2$

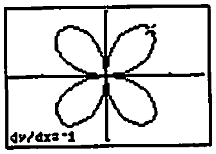
Solve $\frac{dy}{d\theta} = 0$ for $\theta$:

the solutions are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

Using the first, third, and fourth solutions to find $y = r\sin\theta$ reveals horizontal tangent lines at $y = \frac{1}{2}$ and $y = -4$.

Solve $\frac{dx}{d\theta} = 0$ for $\theta$ (by first using the quadratic formula to find $\sin\theta$): the solutions are $\frac{\pi}{2}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Using the last two solutions to find $x = r\cos\theta$ reveals vertical tangent lines at $x = \pm\frac{3\sqrt{3}}{2} \approx \pm 2.598$. Where $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both equal zero $\left(\theta = \frac{\pi}{2}\right)$, inspection of the plot shows that the tangent line is vertical, with equation $x = 0$. (This can be confirmed using L'Hôpital's rule.)

**25.**




$[-1.5, 1.5]$ by $[-1, 1]$

The tips have Cartesian coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. From the curve's symmetries, it is evident that the tangent lines at those points have slopes of $-1$, 1, $-1$, and 1, respectively.

So the equations of the tangent lines are

$y - \frac{1}{\sqrt{2}} = -\left(x - \frac{1}{\sqrt{2}}\right)$ or

$y = -x + \sqrt{2}$,

$y - \frac{1}{\sqrt{2}} = x + \frac{1}{\sqrt{2}}$ or

$y = x + \sqrt{2}$,

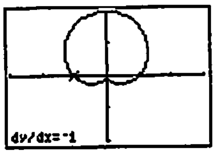
$y + \frac{1}{\sqrt{2}} = -\left(x + \frac{1}{\sqrt{2}}\right)$ or

$y = -x - \sqrt{2}$, and

$y + \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$ or

$y = x - \sqrt{2}$

**26.**




$[-3, 3]$ by $[-2, 2]$

As the plot shows, the curve crosses the $x$-axis at $(x, y)$-coordinates $(-1, 0)$ and $(1, 0)$, with slope $-1$ and 1, respectively. (This can be confirmed analytically.) So the equations of the tangent lines are

$y - 0 = -(x + 1)$

$y = -x - 1$ and

$y - 0 = x - 1$

$y = x - 1$.

**27.** $r\cos\theta = r\sin\theta$

$x = y$, a line

**28.** $r = 3\cos\theta$

$r^2 = 3r\cos\theta$

$x^2 + y^2 = 3x$

$x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4}$

$\left(x - \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$

a circle $\left(\text{center} = \left(\frac{3}{2}, 0\right), \text{radius} = \frac{3}{2}\right)$

**29.** $r = 4 \tan\theta \sec\theta$

$$r\cos\theta = 4\frac{r\sin\theta}{r\cos\theta}$$

$x = 4\frac{y}{x}$ or $x^2 = 4y$, a parabola

**30.** $r\cos\left(\theta + \frac{\pi}{3}\right) = 2\sqrt{3}$

$$r\cos\theta\cos\left(\frac{\pi}{3}\right) - r\sin\theta\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$$

$$\frac{1}{2}r\cos\theta - \frac{\sqrt{3}}{2}r\sin\theta = 2\sqrt{3}$$

$$\frac{1}{2}x - \frac{\sqrt{3}}{2}y = 2\sqrt{3}$$

$x - \sqrt{3}y = 4\sqrt{3}$ or $y = \frac{x}{\sqrt{3}} - 4$, a line

**31.** $x^2 + y^2 + 5y = 0$
$r^2 + 5r\sin\theta = 0$
$r = -5\sin\theta$

**32.** $x^2 + y^2 - 2y = 0$
$r^2 - 2r\sin\theta = 0$
$r = 2\sin\theta$

**33.** $x^2 + 4y^2 = 16$

$$(r\cos\theta)^2 + 4(r\sin\theta)^2 = 16$$

$r^2\cos^2\theta + 4r^2\sin^2\theta = 16$, or $r^2 = \frac{16}{\cos^2\theta + 4\sin^2\theta}$

**34.** $(x + 2)^2 + (y - 5)^2 = 16$
$(r\cos\theta + 2)^2 + (r\sin\theta - 5)^2 = 16$

**35.** $\frac{dx}{dt} = 2e^{2t} - \frac{1}{8}, \frac{dy}{dt} = e^t$, so

$$\text{Length} = \int_0^{\ln 2}\sqrt{\left(2e^{2t} - \frac{1}{8}\right)^2 + (e^t)^2}\,dt$$
$$= \int_0^{\ln 2}\sqrt{4e^{4t} - \frac{1}{2}e^{2t} + \frac{1}{64} + e^{2t}}\,dt$$
$$= \int_0^{\ln 2}\sqrt{\left(2e^{2t} + \frac{1}{8}\right)^2}\,dt$$
$$= \left[e^{2t} + \frac{t}{8}\right]_0^{\ln 2}$$
$$= 4 + \frac{\ln 2}{8} - 1$$
$$= 3 + \frac{\ln 2}{8} = \frac{\ln 2 + 24}{8} \approx 3.087.$$

**36.** $\frac{dx}{dt} = 2t, \frac{dy}{dt} = t^2 - 1$, so

$$\text{Length} = \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{(2t)^2 + (t^2 - 1)^2}\,dt$$
$$= \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{4t^2 + t^4 - 2t^2 + 1}\,dt$$
$$= \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{(t^2 + 1)^2}\,dt$$
$$= \left[\left(\frac{t^3}{3}\right) + t\right]_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3}.$$

**37.** $\frac{dr}{d\theta} = -\sin\theta$, so

$$\text{Length} = \int_0^{2\pi}\sqrt{(-1 + \cos\theta)^2 + (-\sin\theta)^2}\,d\theta$$
$$= \int_0^{2\pi}\sqrt{2 - 2\cos\theta}\,d\theta$$
$$= \int_0^{2\pi}\sqrt{4\sin^2\frac{\theta}{2}}\,d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 8.$$

**38.** $\frac{dr}{d\theta} = 2\cos\theta - 2\sin\theta$, so Length

$$= \int_0^{\pi/2}\sqrt{(2\sin\theta + 2\cos\theta)^2 + (2\cos\theta - 2\sin\theta)^2}\,d\theta$$
$$= \int_0^{\pi/2}\sqrt{8\sin^2\theta + 8\cos^2\theta}\,d\theta$$
$$= \int_0^{\pi/2}2\sqrt{2}\,d\theta = \pi\sqrt{2}.$$

**39.** $\frac{dr}{d\theta} = 8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)$, so

$$\text{Length} = \int_0^{\pi/4}\sqrt{\left(8\sin^3\left(\frac{\theta}{3}\right)\right)^2 + \left(8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right)^2}\,d\theta$$
$$= \int_0^{\pi/4}8\sin^2\left(\frac{\theta}{3}\right)\sqrt{\sin^2\left(\frac{\theta}{3}\right) + \cos^2\left(\frac{\theta}{3}\right)}\,d\theta$$
$$= \int_0^{\pi/4}8\sin^2\left(\frac{\theta}{3}\right)d\theta$$
$$= 8\left[\frac{1}{2}\theta - \frac{3}{4}\sin\left(\frac{2\theta}{3}\right)\right]_0^{\pi/4} = \pi - 3$$

**40.** $\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\sqrt{1 + \cos 2\theta}}$, so Length

$$= \int_{-\pi/2}^{\pi/2}\sqrt{(\sqrt{1 + \cos 2\theta})^2 + \left(\frac{-\sin 2\theta}{\sqrt{1 + \cos 2\theta}}\right)^2}\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\sqrt{\frac{(1 + \cos 2\theta)^2}{1 + \cos 2\theta} + \frac{\sin^2 2\theta}{1 + \cos 2\theta}}\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\sqrt{\frac{1 + 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}{1 + \cos 2\theta}}\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\sqrt{2}\,d\theta = \pi\sqrt{2}$$

**41.** $\frac{dx}{dt} = -2\sin t, \frac{dy}{dt} = 2t$, so

$$\text{Length} = \int_0^{\pi/2}\sqrt{(-2\sin t)^2 + (2t)^2}\,dt$$
$$= \int_0^{\pi/2}2\sqrt{t^2 + \sin^2 t}\,dt,$$

which using NINT evaluates to $\approx 3.183$.

**42.** $\frac{dx}{dt} = 3\cos t, \frac{dy}{dt} = 3\sqrt{t}$, so

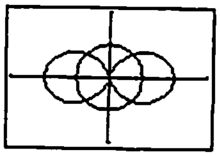
$$\text{Length} = \int_0^3\sqrt{(3\cos t)^2 + (3\sqrt{t})^2}\,dt = \int_0^3 3\sqrt{t + \cos^2 t}\,dt,$$

which using NINT evaluates to $\approx 12.363$.

**43.** $$\text{Area} = \int_0^{2\pi}\frac{1}{2}(2 - \cos\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}(4 - 4\cos\theta + \cos^2\theta)\,d\theta$$
$$= \frac{1}{2}\left[4\theta - 4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{9\pi}{2}$$

44. Area $= \int_0^{\pi/3} \frac{1}{2}\sin^2 3\theta\, d\theta = \frac{1}{2}\left[\frac{1}{2}\theta - \frac{1}{12}\sin(6\theta)\right]_0^{\pi/3} = \frac{\pi}{12}$

45.

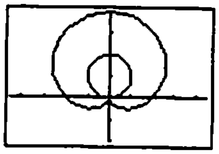


$[-3, 3]$ by $[-2, 2]$

The curves cross where $\cos 2\theta = 0$, such as $\theta = \frac{\pi}{4}$. Using the curves' symmetry,

$$\text{Length} = 4\int_0^{\pi/4} \frac{1}{2}[(1 + \cos 2\theta)^2 - 1]\, d\theta$$
$$= 2\int_0^{\pi/4} (\cos^2 2\theta + 2\cos 2\theta)\, d\theta$$
$$= 2\left[\frac{1}{8}\sin 4\theta + \frac{1}{2}\theta + \sin 2\theta\right]_0^{\pi/4}$$
$$= \frac{\pi}{4} + 2$$

46.



$[-4.5, 4.5]$ by $[-2, 4]$

Since the two curves are covered over different $\theta$-intervals, find the two areas separately. Then

$$\text{Area} = \int_0^{2\pi} \frac{1}{2}[2(1 + \sin\theta)]^2\, d\theta - \pi r^2$$
$$= 2\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta)\, d\theta - \pi$$
$$= 2\left[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} - \pi = 5\pi$$

47. $\frac{dx}{dt} = t,\ \frac{dy}{dt} = 2$, so

$$\text{Area} = \int_0^{\sqrt{5}} 2\pi(2t)\sqrt{t^2 + 2^2}\, dt$$
$$= \left[\frac{4\pi}{3}(t^2 + 4)^{3/2}\right]_0^{\sqrt{5}} = \frac{76\pi}{3}$$

48. $\frac{dx}{dt} = 2t - \frac{1}{2t^2},\ \frac{dy}{dt} = 4$, so

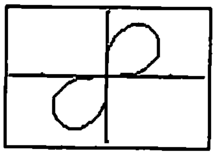
$$\text{Area} = \int_{1/\sqrt{2}}^{1} 2\pi\left(t^2 + \frac{1}{2t}\right)\sqrt{\left(2t - \frac{1}{2t^2}\right)^2 + 4^2}\, dt,$$

which using NINT evaluates to $\approx 10.110$.

49. $\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}$, so

$$\text{Area} = \int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$
$$= \int_0^{\pi/4} 2\pi\sin\theta\, d\theta$$
$$= 2\pi\left[-\cos\theta\right]_0^{\pi/4} = \pi(2 - \sqrt{2}) \approx 1.840$$

50.



$[-1.5, 1.5]$ by $[-1, 1]$

$r = \pm\sqrt{\sin 2\theta}$ and $\frac{dr}{d\theta} = \pm\frac{\cos 2\theta}{\sqrt{\sin 2\theta}}$, where $0 \le \theta \le \frac{\pi}{2}$, so

$$\text{Area} = 2\int_0^{\pi/2} 2\pi\sqrt{\sin 2\theta}\cos\theta\sqrt{\sin 2\theta + \frac{\cos^2 2\theta}{\sin 2\theta}}\, d\theta$$
$$= 4\pi\int_0^{\pi/2} \cos\theta\, d\theta$$
$$= 4\pi\left[\sin\theta\right]_0^{\pi/2} = 4\pi$$

51. (a) $\mathbf{v}(t) = \frac{d}{dt}[(4\cos t)\mathbf{i} + (\sqrt{2}\sin t)\mathbf{j}]$

$$= (-4\sin t)\mathbf{i} + (\sqrt{2}\cos t)\mathbf{j}$$

$\mathbf{a}(t) = \frac{d}{dt}[(-4\sin t)\mathbf{i} + (\sqrt{2}\cos t)\mathbf{j}]$

$$= (-4\cos t)\mathbf{i} + (-\sqrt{2}\sin t)\mathbf{j}$$

(b) $\left|\mathbf{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{\left(-4\sin\frac{\pi}{4}\right)^2 + \left(\sqrt{2}\cos\frac{\pi}{4}\right)^2}$

$$= \sqrt{8 + 1} = 3$$

(c) At $t = \frac{\pi}{4}$, $\mathbf{v} = -2\sqrt{2}\mathbf{i} + \mathbf{j}$, $\mathbf{a} = -2\sqrt{2}\mathbf{i} - \mathbf{j}$, and

$$\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|}$$
$$= \cos^{-1}\frac{8 - 1}{(3)(3)}$$
$$= \cos^{-1}\frac{7}{9} \approx 38.94°.$$

52. (a) $\mathbf{v}(t) = \frac{d}{dt}[(\sqrt{3}\sec t)\mathbf{i} + (\sqrt{3}\tan t)\mathbf{j}]$

$$= (\sqrt{3}\sec t\tan t)\mathbf{i} + (\sqrt{3}\sec^2 t)\mathbf{j}$$

$\mathbf{a}(t) = \frac{d}{dt}[(\sqrt{3}\sec t\tan t)\mathbf{i} + (\sqrt{3}\sec^2 t)\mathbf{j}]$

$$= \sqrt{3}(\sec t\tan^2 t + \sec^3 t)\mathbf{i} + (2\sqrt{3}\sec^2 t\tan t)\mathbf{j}$$

(b) $|\mathbf{v}(0)| = \sqrt{3\sec^2 0\tan^2 0 + 3\sec^4 0} = \sqrt{0 + 3} = \sqrt{3}$

(c) At $t = 0$, $\mathbf{v} = \sqrt{3}\mathbf{j}$, $\mathbf{a} = \sqrt{3}\mathbf{i}$

$$\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} = \frac{0 + 0}{(\sqrt{3})(\sqrt{3})} = \cos^{-1} 0 = 90°$$

53. $\mathbf{v}(t) = -\frac{t}{(1 + t^2)^{3/2}}\mathbf{i} + \frac{1}{(1 + t^2)^{3/2}}\mathbf{j}$

$$\left|\frac{d\mathbf{r}}{dt}\right| = |\mathbf{v}(t)| = \sqrt{\left(-\frac{t}{(1 + t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1 + t^2)^{3/2}}\right)^2} = \frac{1}{1 + t^2},$$

which is at a maximum of 1 when $t = 0$.

**54.** $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j}$

$\mathbf{a}(t) = \frac{dx}{dt} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i}$

$\quad + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j}$

$\quad = (-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{a}(t) = (e^t \cos t)(-2e^t \sin t) + (e^t \sin t)(2e^t \cos t) = 0$

for all $t$. The angle between $\mathbf{r}$ and $\mathbf{a}$ is always 90°.

**55.** $\left(\int_0^1 (3 + 6t)\, dt\right)\mathbf{i} + \left(\int_0^1 6\pi \cos \pi t\, dt\right)\mathbf{j}$

$= \Big[3t + 3t^2\Big]_0^1 \mathbf{i} + \Big[6 \sin \pi t\Big]_0^1 \mathbf{j} = 6\mathbf{i}$

**56.** $\left(\int_e^{e^2} \frac{2 \ln t}{t}\, dt\right)\mathbf{i} + \left(\int_e^{e^2} \frac{1}{t \ln t}\, dt\right)\mathbf{j}$

$= \Big[\ln^2 t\Big]_e^{e^2} \mathbf{i} + \Big[\ln (\ln t)\Big]_e^{e^2} \mathbf{j} = 3\mathbf{i} + (\ln 2)\mathbf{j}$

**57.** $\mathbf{r}(t) = \int \frac{d\mathbf{r}}{dt}\, dt = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{j}$, so $\mathbf{C} = -\mathbf{i} + \mathbf{j}$, and

$\mathbf{r}(t) = (\cos t - 1)\mathbf{i} + (\sin t + 1)\mathbf{j}$

**58.** $\mathbf{r}(t) = \int \frac{d\mathbf{r}}{dt}\, dt = (\tan^{-1} t)\mathbf{i} + \sqrt{t^2 + 1}\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{j} + \mathbf{C} = \mathbf{i} + \mathbf{j}$, so $\mathbf{C} = \mathbf{i}$ and

$\mathbf{r}(t) = (\tan^{-1} t + 1)\mathbf{i} + \sqrt{t^2 + 1}\mathbf{j}$

**59.** $\frac{d\mathbf{r}}{dt} = \int \frac{d^2\mathbf{r}}{dt^2}\, dt = 2t\mathbf{j} + \mathbf{C}_1$, $\mathbf{r}(t) = \int \frac{d\mathbf{r}}{dt}\, dt = t^2\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$

$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1 = \mathbf{0}$, so $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{C}_2$. And $\mathbf{r}(0) = \mathbf{C}_2 = \mathbf{i}$, so

$\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j}$

**60.** $\frac{d\mathbf{r}}{dt} = \int \frac{d^2\mathbf{r}}{dt^2}\, dt = (-2t)\mathbf{i} + (-2t)\mathbf{j} + \mathbf{C}_1$,

$\mathbf{r}(t) = \int \frac{d\mathbf{r}}{dt}\, dt = -t^2\mathbf{i} - t^2\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$

$\left.\frac{d\mathbf{r}}{dt}\right|_{t=1} = -2\mathbf{i} - 2\mathbf{j} + \mathbf{C}_1 = 4\mathbf{i}$, so $\mathbf{C}_1 = 6\mathbf{i} + 2\mathbf{j}$ and

$\mathbf{r}(t) = (-t^2 + 6t)\mathbf{i} + (-t^2 + 2t)\mathbf{j} + \mathbf{C}_2$

$\mathbf{r}(1) = 5\mathbf{i} + \mathbf{j} + \mathbf{C}_2 = 3\mathbf{i} + 3\mathbf{j}$, so $\mathbf{C}_2 = -2\mathbf{i} + 2\mathbf{j}$, and

$\mathbf{r}(t) = (-t^2 + 6t - 2)\mathbf{i} + (-t^2 + 2t + 2)\mathbf{j}$

**61.** (a) $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle -\frac{3\pi}{4} \sin \frac{\pi}{4}t, \frac{5\pi}{4} \cos \frac{\pi}{4}t \right\rangle$,

$\mathbf{v}(3) = \left\langle -\frac{3\pi}{4\sqrt{2}}, -\frac{5\pi}{4\sqrt{2}} \right\rangle$, and

$|\mathbf{v}(3)| = \sqrt{\frac{9\pi^2}{32} + \frac{25\pi^2}{32}} = \frac{\pi\sqrt{34}}{4\sqrt{2}} = \frac{\pi\sqrt{17}}{4} \approx 3.238$

(b) $x$-component: $\left.\frac{d^2x}{dt^2}\right|_{t=3} = -\frac{3\pi^2}{16} \cos\left(\frac{\pi}{4} \cdot 3\right) = \frac{3\pi^2}{16\sqrt{2}}$

$y$-component: $\left.\frac{d^2y}{dt^2}\right|_{t=3} = \frac{-5\pi^2}{16} \sin\left(\frac{\pi}{4} \cdot 3\right) = -\frac{5\pi^2}{16\sqrt{2}}$

(c) $\frac{x}{3} = \cos \frac{\pi}{4}t$ and $\frac{y}{5} = \sin \frac{\pi}{4}t$, so $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$ or

$\frac{x^2}{9} + \frac{y^2}{25} = 1.$

**62.** (a) $\frac{dx}{dt} = \frac{1}{2}$ and $\frac{dy}{dt} = 5 - t$ so

Length $= \int_0^{10} \sqrt{\left(\frac{1}{2}\right)^2 + (5 - t)^2}\, dt$, which using NINT

evaluates to $\approx 25.874$.

(b) Volume $= \int_0^{10} \pi y^2 \frac{dx}{dt}\, dt$

$= \frac{\pi}{2}\int_0^{10} \left(\frac{t(10 - t)}{2}\right)^2 dt$

$= \frac{\pi}{8}\int_0^{10} (100t^2 - 20t^3 + t^4)\, dt$

$= \frac{\pi}{8}\left[\frac{100}{3}t^3 - 5t^4 + \frac{1}{5}t^5\right]_0^{10} = \frac{1250\pi}{3}$

(c) Area $= \int_0^{10} 2\pi \frac{t(10 - t)}{2} \sqrt{\left(\frac{1}{2}\right)^2 + (5 - t)^2}\, dt$, which

using NINT evaluates to $\approx 1040.728$.

**63.** (a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t \sin t + e^t \cos t}{e^t \cos t - e^t \sin t} = \frac{\cos t + \sin t}{\cos t - \sin t}$

$\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{-1}{-1} = 1$

(b) $\frac{dy}{dt} = e^t(\sin t + \cos t)$, $\frac{dx}{dt} = e^t(\cos t - \sin t)$

$\left(\frac{dy}{dt}\right)^2 = e^{2t}(\sin^2 t + 2 \sin t \cos t + \cos^2 t)$

$= e^{2t}(1 + 2 \sin t \cos t)$

$\left(\frac{dx}{dt}\right)^2 = e^{2t}(\cos^2 t - 2 \cos t \sin t + \sin^2 t)$

$= e^2(1 - 2 \cos t \sin t)$

$|\mathbf{v}(t)| = e^t\sqrt{2}$

$|\mathbf{v}(3)| = e^3\sqrt{2}$

(c) Distance $= \int_0^3 |\mathbf{v}(t)|\, dt$

$= \int_0^3 e^t\sqrt{2}$

$= \sqrt{2}\Big[e^t\Big]_0^3$

$= (e^3 - 1)\sqrt{2}$

**64. (a)** $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 2t, \frac{6}{5}t^2 \right\rangle$, $\mathbf{v}(4) = \left\langle 8, \frac{96}{5} \right\rangle$, and

$|\mathbf{v}(4)| = \sqrt{8^2 + \left(\frac{96}{5}\right)^2} = \frac{104}{5}$

**(b)** Distance $= \int_0^4 \sqrt{(2t)^2 + \left(\frac{6}{5}t^2\right)^2}\, dt$

$= \int_0^4 \frac{2}{5}t\sqrt{25 + 9t^2}\, dt$

$= \left[\frac{2}{135}(25 + 9t^2)^{3/2}\right]_0^4 = \frac{4144}{135}$

**(c)** $t = \sqrt{x + 2}$, so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2/5}{2t} = \frac{3}{5}t = \frac{3}{5}\sqrt{x + 2}$

**65.** $x$ degrees east of north is $(90 - x)$ degrees north of east.

Add the vectors:

$\langle 540 \cos 10°, 540 \sin 10° \rangle + \langle 55 \cos (-10°), 55 \sin (-10°) \rangle$

$= \langle 595 \cos 10°, 485 \sin 10° \rangle$

$\approx \langle 585.961, 84.219 \rangle$.

Speed $\approx \sqrt{585.961^2 + 84.219^2} \approx 591.982$ mph.

Direction $\approx \tan^{-1}\left(\frac{585.961}{84.219}\right) \approx 81.821°$ east of north

**66.** Add the vectors:

$\langle 120 \cos 20°, 120 \sin 20° \rangle + \langle 300 \cos (-5°), 300 \sin (-5°) \rangle$

$\approx \langle 411.622, 14.896 \rangle$.

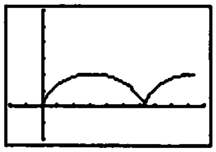
Direction $\approx \tan^{-1}\left(\frac{14.896}{411.622}\right) \approx 2.073°$

Length $\approx \sqrt{411.622^2 + 14.896^2} \approx 411.891$ lbs

**67.** Taking the launch point as the origin, $y = (44 \sin 45°)t - 16t^2$ equals $-6.5$ when $t \approx 2.135$ sec (as can be determined graphically or using the quadratic formula). Then $x \approx (44 \cos 45°)(2.135) \approx 66.421$ horizontal feet from where it left the thrower's hand. Assuming it doesn't bounce or roll, it will still be there 3 seconds after it was thrown.

**68.** $y_{max} = \frac{(80 \sin 45°)^2}{2(32)} + 7 = 57$ feet

**69. (a)**



[−2, 10] by [−2, 6]

**(b)** $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle \pi - \pi \cos \pi t, \pi \sin \pi t \rangle$

$\mathbf{a}(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t \rangle$

$\mathbf{v}(0) = \langle 0, 0 \rangle$ $\quad$ $\mathbf{v}(1) = \langle 2\pi, 0 \rangle$
$\mathbf{a}(0) = \langle 0, \pi^2 \rangle$ $\quad$ $\mathbf{a}(1) = \langle 0, -\pi^2 \rangle$
$\mathbf{v}(2) = \langle 0, 0 \rangle$ $\quad$ $\mathbf{v}(3) = \langle 2\pi, 0 \rangle$
$\mathbf{a}(2) = \langle 0, \pi^2 \rangle$ $\quad$ $\mathbf{a}(3) = \langle 0, -\pi^2 \rangle$

**(c)** Topmost point: $2\pi$ ft/sec
center of wheel: $\pi$ ft/sec
Reasons: Since the wheel rolls half a circumference, or $\pi$ feet every second, the center of the wheel will move $\pi$ feet every second. Since the rim of the wheel is turning at a rate of $\pi$ ft/sec about the center, the velocity of the topmost point relative to the center is $\pi$ ft/sec, giving it a total velocity of $2\pi$ ft/sec.

**70.** $v_0 = \sqrt{\frac{Rg}{\sin 2\alpha}}$, where $\alpha = 45°$, $g = 32$, and $R$ = range

for 4325 yds = 12,975 ft: $v_0 \approx 644.360$ ft/sec

for 4752 yds = 14,256 ft: $v_0 \approx 675.420$ ft/sec

**71. (a)** $v_0 = \sqrt{\frac{Rg}{\sin 2\alpha}} = \sqrt{(109.5)(32)} \approx 59.195$ ft/sec

**(b)** The cork lands at $y = -4$, $x = 177.75$.

Solve $y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x$ for $v_0$, with $\alpha = 45°$: $v_0 = \sqrt{-\frac{gx^2}{y - x}} \approx 74.584$ ft/sec

**72. (a)** The javelin lands at $y = -6.5$, $x = 262\frac{5}{12}$.

Solve $y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x$ for $v_0$, with $\alpha = 40°$:

$v_0 = \sqrt{-\frac{gx^2}{(2 \cos^2 40°)(y - x \tan 40°)}} \approx 91.008$ ft/sec

**(b)** $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} + 6.5$

$\approx \frac{(91.008 \sin 40°)^2}{64} + 6.5 \approx 59.970$ ft

**73.** We have $x = (v_0 t) \cos \alpha$ and

$y + \frac{gt^2}{2} = (v_0 t) \sin \alpha$. Squaring and adding gives

$x^2 + \left(y + \frac{gt^2}{2}\right)^2 = (v_0 t)^2(\cos^2 \alpha + \sin^2 \alpha) = v_0^2 t^2$.

74. (a) $\mathbf{r}(t) = (155 \cos 18° - 11.7)t\mathbf{i} + (4 + 155 \sin 18°t - 16t^2)\mathbf{j}$

$x(t) = (155 \cos 18° - 11.7)t$

$y(t) = 4 + 155 \sin 18°t - 16t^2$

(b) $y_{max} = \frac{(155 \sin 18°)^2}{2(32)} + 4 \approx 39.847$ feet, reached at

$t_{max} = \frac{155 \sin 18°}{32} \approx 1.497$ sec

(c) $y(t) = 0$ when $t \approx 3.075$ sec (found using the quadratic formula), and then $x \approx (155 \cos 18° - 11.7)(3.075) \approx 417.307$ ft.

(d) Solve $y(t) = 25$ using the quadratic formula:

$$t = \frac{-155 \sin 18° \pm \sqrt{155^2 \sin^2 18° - 4(16)(21)}}{-32}$$

$\approx 0.534$ and $2.460$ seconds.

At those times, $x = (155 \cos 18° - 11.7)t$ equals

$\approx 72.406$ and $\approx 333.867$ feet from home plate.

(e) Yes, the batter has hit a home run. When the ball is 380 feet from home plate (at $t \approx 2.800$ seconds), it is approximately 12.673 feet off the ground and therefore clears the fence by at least two feet.

75. (a) $$\mathbf{r}(t) = \left[(155 \cos 18° - 11.7)\frac{1}{0.09}(1 - e^{-0.09t})\right]\mathbf{i} + \left[4 + \left(\frac{155 \sin 18°}{0.09}\right)(1 - e^{-0.09t}) + \frac{32}{0.09^2}(1 - 0.09t - e^{-0.09t})\right]\mathbf{j}$$

$$x(t) = (155 \cos 18° - 11.7)\frac{1}{0.09}(1 - e^{-0.09t})$$

$$y(t) = 4 + \left(\frac{155 \sin 18°}{0.09}\right)(1 - e^{-0.09t}) + \frac{32}{0.09^2}(1 - 0.09t - e^{-0.09t})$$

(b) Plot $y(t)$ and use the maximum function to find $y \approx 36.921$ feet at $t \approx 1.404$ seconds.

(c) Plot $y(t)$ and find that $y(t) = 0$ at $t \approx 2.959$, then plug this into the expression for $x(t)$ to find $x(2.959) \approx 352.520$ ft.

(d) Plot $y(t)$ and find that $y(t) = 30$ at $t \approx 0.753$ and 2.068 seconds. At those times, $x \approx 98.799$ and 256.138 feet (from home plate).

(e) No, the batter has not hit a home run. If the drag coefficient $k$ is less than $\approx 0.011$, the hit will be a home run.
(This result can be found by trying different $k$-values until the parametrically plotted curve has $y \geq 10$ for $x = 380$.)

76. (a) $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

(b) $\overrightarrow{AP} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$

(c) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$, so by part (b), $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$.

77. The widths between the successive turns are constant and are given by $2\pi a$.

# Cumulative Review Exercises (pp. 573–576)

1. Since the function has no discontinuity at $x = 1$, the limit is

$$\frac{2(1)^2 - 1 - 1}{1^2 + 1 - 12} = 0.$$

2. By l'Hôpital's Rule, $\lim_{x\to 0} \frac{\sin 3x}{4x} = \lim_{x\to 0} \frac{3 \cos 3x}{4} = \frac{3}{4}$.

3. By l'Hôpital's Rule, $\lim_{x\to 0} \frac{\frac{1}{x+1} - 1}{x} = \lim_{x\to 0} \frac{-\frac{1}{(x+1)^2}}{1} = -1$.

4. By l'Hôpital's Rule, $\lim_{x\to\infty} \frac{x + e^x}{x - e^x} = \lim_{x\to\infty} \frac{1 + e^x}{1 - e^x}$

$= \lim_{x\to\infty} \frac{e^x}{-e^x} = -1$.

5. By l'Hôpital's Rule, $\lim_{t\to 0} \frac{t(1 - \cos t)}{t - \sin t}$

$= \lim_{t\to 0} \frac{t \sin t + (1 - \cos t)}{1 - \cos t} = \lim_{t\to 0} \frac{t \cos t + 2 \sin t}{\sin t}$

$= \lim_{t\to 0} \frac{-t \sin t + 3 \cos t}{\cos t} = 3$

6. By l'Hôpital's Rule, $\lim_{x\to 0^+} \frac{\ln (e^x - 1)}{\ln x} = \lim_{x\to 0^+} \frac{\frac{e^x}{(e^x - 1)}}{\frac{1}{x}}$

$= \lim_{x\to 0^+} \frac{xe^x}{e^x - 1} = \lim_{x\to 0^+} \frac{xe^x + e^x}{e^x} = 1$

7. Use $f(x) = (e^x + x)^{1/x}$. Then $\ln f(x) = \frac{\ln (e^x + x)}{x}$, and

$\lim_{x\to 0} \frac{\ln (e^x + x)}{x} = \lim_{x\to 0} \frac{(e^x + 1)/(e^x + x)}{1} = 2.$

So $\lim_{x\to 0} (e^x + x)^{1/x} = \lim_{x\to 0} e^{\ln f(x)} = e^2$.

8. $\lim_{x\to 0} \left(\frac{3x + 1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{(3x + 1) \sin x - x}{x \sin x}$

$= \lim_{x\to 0} \frac{(3x + 1) \cos x + 3 \sin x - 1}{x \cos x + \sin x}$

$= \lim_{x\to 0} \frac{-(3x + 1) \sin x + 6 \cos x}{-x \sin x + 2 \cos x} = 3$

**9.** (a) $2(1) - 1^2 = 1$

(b) $2 - 1 = 1$

(c) 1 [from (a) and (b)]

(d) Yes, since $\lim_{x \to 1} f(x) = f(1) = 1$

(e) No.

Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2(1+h) - (1+h)^2 - 1}{h}$$
$$= \lim_{h \to 0^-} \frac{2 + 2h - 1 - 2h - h^2 - 1}{h}$$
$$= \lim_{h \to 0^-} -\frac{h^2}{h}$$
$$= \lim_{h \to 0^-} -h = 0$$

Right-hand derivative:

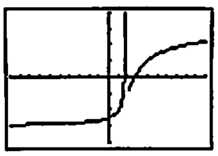
$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2 - (1+h) - 1}{h}$$
$$= \lim_{h \to 0^+} \frac{-h}{h} = -1$$

Since the left- and right-hand derivatives are not equal, $f$ is not differentiable at $x = 1$.

**10.** Solve $4 - x^2 \le 0$: all $x \le -2$ and $x \ge 2$.

**11.** Horizontal: since as $x \to \pm\infty$, $2x^2 - x \to +\infty$ while $-1 \le \cos x \le 1$, the end behavior at both ends is $y = 0$. Vertical: solve $2x^2 - x = 0$ to find $x = 0, x = \frac{1}{2}$.

**12.** One possible function is $y = \begin{cases} -3 + \dfrac{1}{(2-x)}, & x < 2 \\ 3 - \dfrac{8}{x}, & x \ge 2 \end{cases}$



[−10, 10] by [−4, 4]

**13.** $\dfrac{f(5) - f(0)}{5 - 0} = \dfrac{\sqrt{9} - \sqrt{4}}{5} = \dfrac{1}{5}$

**14.** $y' = \dfrac{(x-2)(1) - (x+1)(1)}{(x-2)^2} = -\dfrac{3}{(x-2)^2}$

**15.** $y' = -\sin(\sqrt{1-3x})\left[\frac{1}{2}(1-3x)^{-1/2}\right](-3)$

$$= \frac{3 \sin \sqrt{1-3x}}{2\sqrt{1-3x}}$$

**16.** $y' = \sin x \sec^2 x + \tan x \cos x = \dfrac{\sin x}{\cos^2 x} + \sin x$

$$= \frac{(\sin x)(1 + \cos^2 x)}{\cos^2 x}$$

**17.** $y' = \left(\dfrac{1}{x^2+1}\right)(2x) = \dfrac{2x}{x^2+1}$

**18.** $y' = (e^{x^2-x})(2x-1) = (2x-1)e^{x^2-x}$

**19.** $y' = 2x \tan^{-1} x + \dfrac{x^2}{1+x^2}$

**20.** $y' = -3x^{-4}e^x + e^x x^{-3} = (x^{-3} - 3x^{-4})e^x$

**21.** $y' = 3\left(\dfrac{\csc x}{1 + \cos x}\right)^2 \left(\dfrac{(1 + \cos x)(-\csc x \cot x) + \csc x \sin x}{(1+\cos x)^2}\right)$

$$= \frac{3 \csc^2 x}{(1 + \cos x)^4}(1 - \csc x \cot x - \cos x \csc x \cot x)$$
$$= \frac{3 \csc^2 x}{(1 + \cos x)^4}(1 - \cot x \csc x - \cot^2 x)$$
$$= \frac{3 \csc^2 x}{(1 + \cos x)^4}(1 - \csc^2 x + \csc^2 x - \cot x \csc x - \cot^2 x)$$
$$= \frac{3 \csc^2 x}{(1 + \cos x)^4}(\csc^2 x - \cot x \csc x - 2\cot^2 x)$$
$$= \left(\frac{3}{(\sin^2 x)(1 + \cos x)^4}\right)\left(\frac{1 - \cos x - 2\cos^2 x}{\sin^2 x}\right)$$
$$= \left(\frac{3}{(\sin^2 x)(1 + \cos x)^4}\right)\left(\frac{(1 + \cos x)(1 - 2\cos x)}{\sin^2 x}\right)$$
$$= \frac{3(1 - 2\cos x)}{(\sin^4 x)(1 + \cos x)^3}$$

**22.** $y' = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \sin^{-1} x\right) - \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \tan^{-1} x\right)$

$$= -\frac{1}{\sqrt{1-x^2}} + \frac{1}{1+x^2}$$

**23.** $\dfrac{d}{dx}[\cos(xy) + y^2 - \ln x] = \dfrac{d}{dx}(0)$

$$-\sin(xy)(xy' + y) + 2yy' - \frac{1}{x} = 0$$
$$y' = \frac{\frac{1}{x} + y\sin(xy)}{-x\sin(xy) + 2y} = \frac{1 + xy\sin(xy)}{2xy - x^2\sin(xy)}$$

**24.** $y' = \dfrac{1}{2}|x|^{-1/2}\dfrac{d}{dx}|x| = \dfrac{1}{2\sqrt{|x|}}\left(\dfrac{|x|}{x}\right) = \dfrac{|x|}{2x\sqrt{|x|}}$

**25.** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-\cos t}{-\sin t} = \cot t = \dfrac{x-1}{1-y}$

**26.** $\ln y = \ln[(\cos x)^x]$

$$\ln y = x \ln(\cos x)$$
$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{\cos x}\right)(-\sin x) + \ln \cos x$$
$$\frac{dy}{dx} = y \cdot \left(\ln(\cos x) - \frac{x \sin x}{\cos x}\right)$$
$$= (\cos x)^x\left(\ln(\cos x) - \frac{x \sin x}{\cos x}\right)$$
$$= (\cos x)^{x-1}[\cos x \ln(\cos x) - x \sin x]$$

**27.** By the Fundamental theorem of Calculus, $y' = \sqrt{1 + x^3}$.

**28.** $y = \left[-\cos t\right]_{2x}^{x^2} = -\cos(x^2) + \cos(2x)$;

$$y' = 2x\sin(x^2) - 2\sin(2x)$$