UNIT VI

UNIT VI
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UNIT VI: TECHNIQUES OF INTEGRATION

Expectation: The student will integrate elementary functions using a variety of techniques.

OVERVIEW:

The following techniques of integration will be explored: substitution, integration by parts and methods of partial fractions. Improper integrals will be discussed. Although trigonometric substitution is no longer an AP objective it is included because of its importance in the next level of Calculus.

INDICATORS:

1. Integrate expressions using a change of variable including change of limits for definite integrals.

2. Integrate expressions using the method of integration by parts

3. Integrate rational expressions using the method of partial fractions (nonrepeating linear factors only).

4. Use substitution to integrate expressions of the form: $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, $\sqrt{u^2 - a^2}$, $a^2 + u^2$, $a^2 - u^2$.

5. Identify and evaluate integrals which satisfy the conditions for an improper integral.

6. Evaluate limits with exponential indeterminate form.

* BC Calculus indicator only

Unit VI TECHNIQUES OF INTEGRATION

Indicators/ Objectives	Foerster: Calculus Key Curriculum 1998	Foerster: Calculus: Instructor's Resource Book Key Curriculum 1998	Finney, et al: Calculus S F A W 1999	Guide Pages
1	459		319	VI 1-2
2	436 - 490	9.3	323 - 328	
3	462 - 464	9.7	444 - 446	
4	456 - 460	9.6	449; 315 - 319	
5	484 - 490	9.1	433 - 441	
6	305		421 - 423	

Indicators/ Objectives	Finney, et al: Calculus 1994	Guide Pages
1		
2	620 - 627	
3	642 - 643	
4	635 - 640	
5	648	
6	571 - 572	

AP CALCULUS
UNIT VI – ACTIVITY 1

NAME Key
DATE ____________

INTEGRATION BY SUBSTITUTION USING A CHANGE OF LIMITS

For each integral, use the given substitution, state du and dx, find the new limits of integration, and write the new integral expression. When completed, use the fnInt function on the calculator to check that the value of the original integral is equal to the value of the new integral.

1. $\int_0^3 \frac{6x^2dx}{\sqrt{1+x^3}}$ given $u=1+x^3$ Find:

$du = 3x^2dx$

$\frac{du}{3x^2} = dx$

or: $x^2dx = \frac{1}{3}du$

$6x^2dx = \frac{6}{3}du = 2du$

I: $2\int \frac{du}{u^{1/2}}$

Limits: $0 \le x \le 3$
$1 \le u = 1+x^3 \le 28$

du = $3x^2dx$

dx = $\frac{du}{3x^2}$

new lower limit = 1 (when $x=0$, $u=1+x^3$, so $u=1$)

new upper limit = 28 (when $x=3$, $u=1+x^3 \Rightarrow u=28$)

new integral = $2\int_1^{28} u^{-1/2}du$

check using calculator:

value of original ≅ 17.166 value of new integral ≅ 17.166

2. $\int_0^1 \frac{x\,dx}{(1+x^2)^2}$ given $u=1+x^2$ Find:

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

I: $\frac{1}{2}\int \frac{du}{u^2}$

Limits:
$0 \le x \le 1$
$1 \le u = 1+x^2 \le 2$

du = $2x\,dx$

dx = $\frac{du}{2x}$

new lower limit = 1 (when $x=0$, $u=1$)

new upper limit = 2 (when $x=1$, $u=2$)

new integral = $\frac{1}{2}\int_1^2 u^{-2}du$

check using calculator:

value of original = .25 value of new integral = .25

3. $\int_0^5 \sqrt{25-x^2}\,dx$ given $x = 5\sin\Theta$ Find: dx = $5\cos\theta\,d\theta$

$\Theta = \sin^{-1}\frac{x}{5}$

Limits:
$0 \le x \le 5$
$0 \le \theta = \sin^{-1}\frac{x}{5} \le \pi/2$

$x^2 = 25\sin^2\theta$

$25 - x^2 = 25 - 25\sin^2\theta$

$= 25\cos^2\theta$

$\sqrt{25-x^2} = \sqrt{25\cos^2\theta}$

$= 5\cos\theta$

new lower limit = 0

new upper limit = $\pi/2$

new integral = $\int_0^{\pi/2} 25\cos^2\theta\,d\theta$

$= \frac{25}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$, etc.

check using calculator:

value of original $\doteq$ 19.635 value of new integral $\doteq$ 19.635

4. $\int_0^1 \frac{x^3\,dx}{\sqrt{x^2+1}}$ given $x = \tan\Theta$ Find: dx = $\sec^2\theta\,d\theta$

$\Theta = \tan^{-1}x$

$0 \le x \le 1$

$x = \tan\theta$

$0 \le \theta = \tan^{-1}x \le \pi/4$

new lower limit = 0

new upper limit = $\pi/4$

new integral = $\int_0^{\pi/4} \tan^3\theta\sec\theta\,d\theta$

check using calculator:

value of original $\doteq$ .195 value of new integral $\doteq$ .195

UNIT VI
STUDENT WORKSHEETS

NAME ____________________
DATE ____________________

INTEGRATION BY SUBSTITUTION USING A CHANGE OF LIMITS

For each integral, use the given substitution, state du and dx, find the new limits of integration, and write the new integral expression. When completed, use the fnInt function on the calculator to check that the value of the original integral is equal to the value of the new integral.

1. $\int_0^3 \frac{6x^2dx}{\sqrt{1+x^3}}$ given $u=1+x^3$ Find:

du = ____________________

dx = ____________________

new lower limit = ____________________

new upper limit = ____________________

new integral = ____________________

check using calculator:

value of original = ____________________ value of new integral = ____________________

2. . $\int_0^1 \frac{x\,dx}{(1+x^2)^2}$ given $u=1+x^2$ Find:

du = ____________________

dx = ____________________

new lower limit = ____________________

new upper limit = ____________________

new integral = ____________________

check using calculator:

value of original = ____________________ value of new integral = ____________________

3. $\int_0^5 \sqrt{25 - x^2}\, dx$ given $x = 5\sin\Theta$ Find: dx = ________________

Θ = ________________

new lower limit = ________________

new upper limit = ________________

new integral = ________________

check using calculator:

value of original = ________________ value of new integral = ________________

4. $\int_0^1 \frac{x^3\, dx}{\sqrt{x^2 + 1}}$ given $x = \tan\Theta$ Find: dx = ________________

Θ = ________________

new lower limit = ________________

new upper limit = ________________

new integral = ________________

check using calculator:

value of original = ________________ value of new integral = ________________

NAME ____________________
DATE ____________________

MULTIPLE CHOICE PRACTICE

1. If the substitution $u = \sin 2x$ is made, $\int_0^{\frac{\pi}{4}} \sin^2 2x \cos 2x \, dx =$

a) $\int_0^1 \frac{u^2 \, du}{2}$

b) $\int_0^{\frac{\pi}{4}} \frac{u^2 \, du}{2}$

c) $\int_0^1 2u^2 \, du$

d) $\int_0^{\frac{\pi}{4}} 4u^2 \, du$

e) $\int_0^1 4u^2 \, du$

2. $\int \frac{dx}{x^2 - 4} =$

a) $\frac{1}{2}\ln|(x+2)(x-2)| + c$

b) $\frac{1}{2}\ln\left|\frac{2}{x-2}\right| + c$

c) $\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right| + c$

d) $\frac{1}{4}\ln\left|\frac{x+2}{x-2}\right| + c$

e) $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + c$

3. $\int_2^{\infty} \frac{dx}{(1+x)^2} =$

a) $-\frac{1}{2}$

b) $\frac{1}{2}$

c) $-\frac{1}{3}$

d) $\frac{1}{3}$

e) divergent

4. $\int x^2 \cos 2x \, dx =$

a) $\frac{1}{2}x^2 \sin 2x + \frac{1}{4}x \cos 2x - \frac{1}{8}\sin 2x + c$

b) $\frac{1}{2}x^2 \sin 2x + \frac{1}{2}x \cos 2x - \frac{1}{4}\sin 2x + c$

c) $\frac{1}{2}x^2 \sin 2x + \frac{1}{4}x \cos 2x + \frac{1}{4}\sin 2x + c$

d) $2x^2 \sin 2x - 4x \cos 2x + 4\sin 2x + c$

e) $2x^2 \sin 2x + 2x \cos 2x - 4\sin 2x + c$

AP CALCULUS
UNIT VI

NAME Key
DATE

MULTIPLE CHOICE PRACTICE

1. If the substitution $u = \sin 2x$ is made, $\int_0^{\frac{\pi}{4}} \sin^2 2x \cos 2x \, dx =$

a) $\int_0^1 \frac{u^2 \, du}{2}$

b) $\int_0^{\frac{\pi}{4}} \frac{u^2 \, du}{2}$

c) $\int_0^1 2u^2 \, du$

d) $\int_0^{\frac{\pi}{4}} 4u^2 \, du$

e) $\int_0^1 4u^2 \, du$

(A)

2. $\int \frac{dx}{x^2 - 4} =$

a) $\frac{1}{2}\ln|(x+2)(x-2)| + c$

b) $\frac{1}{2}\ln\left|\frac{2}{x-2}\right| + c$

c) $\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right| + c$

d) $\frac{1}{4}\ln\left|\frac{x+2}{x-2}\right| + c$

e) $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + c$

(E)

3. $\int_2^{\infty} \frac{dx}{(1+x)^2} =$

a) $-\frac{1}{2}$

b) $\frac{1}{2}$

c) $-\frac{1}{3}$

d) $\frac{1}{3}$

e) divergent

(D)

4. $\int x^2 \cos 2x \, dx =$

a) $\frac{1}{2}x^2 \sin 2x + \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x + c$

b) $\frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c$

c) $\frac{1}{2}x^2 \sin 2x + \frac{1}{4}x\cos 2x + \frac{1}{4}\sin 2x + c$

d) $2x^2 \sin 2x - 4x\cos 2x + 4\sin 2x + c$

e) $2x^2 \sin 2x + 2x\cos 2x - 4\sin 2x + c$

(B)

1. Complete the identity using one term. $1+\tan^2 x =$

4. $\lim_{h \to 0} \frac{e^{3+h} - e^3}{h} =$

3. $y = \csc^{-1} x$
   $y' =$

4. $f(x) = \sec(e^{3x})$
   $f'(x) =$

5. Express $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} e^{\sin x} \cos x \, dx$ in terms of $u = \sin x$.

1. Complete the identity using one term.
$1+\tan^2 x =$

2. $\lim_{h\to 0}\dfrac{e^{3+h}-e^{3}}{h}$

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$f'(x) =$

5. Express
$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} e^{\sin x}\cos x\,dx$
in terms of $u = \sin x$.

1. $\sec^2 x$

2. $f(x) = e^x$
$f'(x) = e^x$
$f'(3) = e^3$

3. $\dfrac{-1}{x\sqrt{x^2-1}}$

4. $3\sec(e^{3x})\tan(e^{3x})\,e^{3x}$

5. $\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} e^{u}\,du$

1. $\int_4^7 (x+2)^2\,dx$
Rename as an integral in terms of u = x +2.

2. Complete the identity using first degree expressions:
$\cos^2 x =$

3. $\arcsin(x+y) = x$
$\frac{dy}{dx} =$

4. $f(x) = \ln^6 x$
$f'(x) =$

5. $\int \ln x\,dx =$
*

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5. $\int \ln x\,dx =$
*

1. $\int_6^9 u^2\,du$

2. $\frac{1}{2}(1+\cos x)$

3. $x + y = \sin x$
$y = \sin x - x$
$y' = \cos x - 1$

4. $\frac{6\ln^5 x}{x}$

5. $x\ln x - x$

1. $\int_4^7 (x+2)^2\,dx$
Rename as an integral in terms of u = x +2.

2. Complete the identity using first degree expressions:
$\cos^2 x =$

3. $\arcsin(x+y) = x$
$\frac{dy}{dx} =$

4. $f(x) = \ln^6 x$
$f'(x) =$

5. $\int \ln x\,dx =$
*

1. Write the linearization of
$f(x)=\frac{2}{x}+2$ at $x=2$

2. $\frac{d}{dx}\int_4^{\sqrt{x}} \sin 3t\, dt$

3. State the hypothesis of the Mean Value Theorem.

4. The radius of a ripple on a lake is increasing at a rate of 3 cm/sec. How fast is the area of the ripple increasing when its radius is 7 cm?

5. $x=3+\cos t \quad y=-4+e^t$
* Write an integral expression for the length of the curve from $t=1$ to $t=2$.

1. Write the linearization of $f(x) = \frac{2}{x} + 2$ at $x = 2$

2. $\frac{d}{dx}\int_4^{\sqrt{x}} \sin 3t \, dt$

3. State the hypothesis of the Mean Value Theorem.

4. The radius of a ripple on a lake is increasing at a rate of 3 cm/sec. How fast is the area of the ripple increasing when its radius is 7 cm?

5. $x = 3 + \cos t \quad y = -4 + e^t$
Write an integral expression for the length of the curve from $t = 1$ to $t = 2$.

1. L(x) = 3 - 0.5(x-2)

2. $\frac{\sin 3\sqrt{x}}{2\sqrt{x}}$

4. f(x) is differentiable on (a,b) and continuous on [a,b].

4. $42\pi \frac{cm^2}{\sec}$

5. $\int_1^2 \sqrt{\sin^2 t + e^{2t}}\, d$

Let x = sin θ for #1- 4.

1. What is θ when x is 0.5 ?

2. Simplify $\sqrt{1-x^2}$ in terms of θ.

3. Express dx in terms of θ.

4. Rename $\int_{0.5}^{1} \sqrt{1-x^2}\, dx$ as an integral in terms of θ.

5. Rename with partial
* fractions: $\dfrac{4}{x^2-5x-6}$

Let x = sin θ for #1- 4.

1. What is θ when x is 0.5 ?

2. Simplify $\sqrt{1-x^2}$ in terms of θ.

3. Express dx in terms of θ.

4. Express $\int_{0.5}^{1} \sqrt{1-x^2}\, dx$ as an integral in terms of θ.

5. Rename with partial
* fractions: $\dfrac{4}{x^2-5x-6}$

$\theta = \sin^{-1} x$

1. $\theta = \dfrac{\pi}{6}$

2. $\sqrt{1-\sin^2\theta}$

$= \cos\theta$

3. $dx = \cos\theta d\theta$

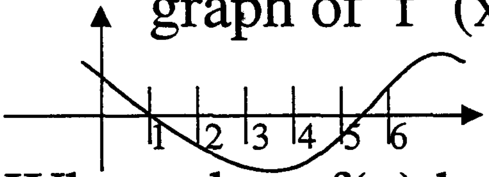
4. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\theta\, d\theta$

5. $\dfrac{4}{7(x-6)} - \dfrac{4}{7(X+1)}$

1.

x	0	1	2	3
f(x)	1.4	1.5	1.7	2.0

Estimate $f'(2)$

2. 

Where does f(x) have a local maximum?

3. $\int 2^x \, dx$

4. If $\int_0^3 f(x)\,dx = 9$
and $f(x)$ is an even function,
evaluate $\int_{-3}^{3} f(x)\,dx$

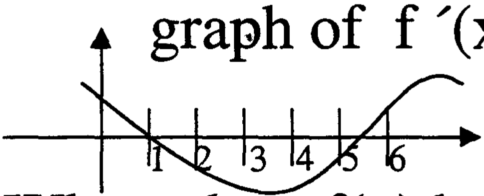
5. $x = \ln t \quad y = \cos^{-1} t$
* $\frac{dy}{dx} =$

1.

x	0	1	2	3
f(x)	1.4	1.5	1.7	2.0

Estimate f ´(2)

2. graph of f ´(x)



Where does f(x) have a local maximum?

3. $\int 2^x \, dx$

4. If $\int_0^3 f(x)\,dx = 9$ and $f(x)$ is an even function, evaluate $\int_{-3}^{3} f(x)\,dx$

5. $x = \ln t \quad y = \cos^{-1} t$

* $\dfrac{dy}{dx} =$

1.

$0.2 \le f'(2) < 0.3$

2. at x = 1

3. $\dfrac{2^x}{\ln 2} + c$

4. 18

5.

$\dfrac{dy}{dt} = -\dfrac{1}{\sqrt{1-t^2}}$

$and \; \dfrac{dx}{dt} = \dfrac{1}{t}$

$\therefore \dfrac{dy}{dx} = \dfrac{-t}{\sqrt{1-t^2}}$

UNIT VI INTERNET RESOURCES

http://www.netsrq.com/~hahn/calculus.html, http://www.barzilai.org/archive, http://archives.math.utk.edu.visual.calculus/ and http://www.hofstra.edu/~matscw/RealWorld/index.html

Sites feature some tutorials, but mostly have good drill and quiz resources for either in-class practice or at-home practice.

http://www.integrals.com

Site features an integral calculator (indefinite integrals) and has a history of integration.