UNIT IX

UNIT IX
TABLE OF CONTENTS

* UNIT IX: POLAR CURVES AND VECTOR VALUE FUNCTIONS

Expectation:	Students will be able to find the area enclosed by a polar curve. They will be able to determine the velocity, speed and acceleration of a particle expressed as a vector.

OVERVIEW:

Extend the concepts of calculus to vector value functions and polar curves.

INDICATORS:

1. Find the area enclosed by a polar curve.

2. Given the coordinates of a particle in terms of a vector, determine the velocity, speed, and acceleration.

* BC Calculus indicators only

Unit IX: Polar Curves and Vector Value Functions

Indicators/ Objectives	Foerster: Calculus Key Curriculum 1998	Foerster: Calculus: Instructor's Resource Book Key Curriculum 1998	Finney, et al: Calculus S F A W 1999	Guide Pages
1	418-419	8.9	561-563	
2	532-540	10.7a,b	533	IX 1-4

Indicators/ Objectives	Finney, et al: Calculus 1994	Guide Pages
1	834 - 837	
2	918	

NAME Key
DATE

PARAMETRIC AND VECTOR VALUE FUNCTIONS

1. If $x = t^2 - 1$ and $y = 2t^3 + 1$, then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2}{2t} = 3t$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)/dt}{dx/dt} = \frac{3}{2t}$$

2. If $x = 4\cos 2\theta$ and $y = 5\sin 3\theta$, then find $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{15\cos 3\theta}{-8\sin 2\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)/d\theta}{dx/d\theta} = -\frac{15}{8} \cdot \frac{\frac{\sin 2\theta \cdot (-3\sin 3\theta) - \cos 3\theta \cdot 2\cos 2\theta}{\sin^2 2\theta}}{-8\sin 2\theta}$$

$$= -\frac{15}{8} \frac{-3\sin 2\theta \sin 3\theta - 2\cos 2\theta \cos 3\theta}{-8\sin^3 2\theta}$$

$$= \frac{15}{64} \frac{-3\sin 2\theta \sin 3\theta - 2\cos 2\theta \cos 3\theta}{\sin^3 2\theta}$$

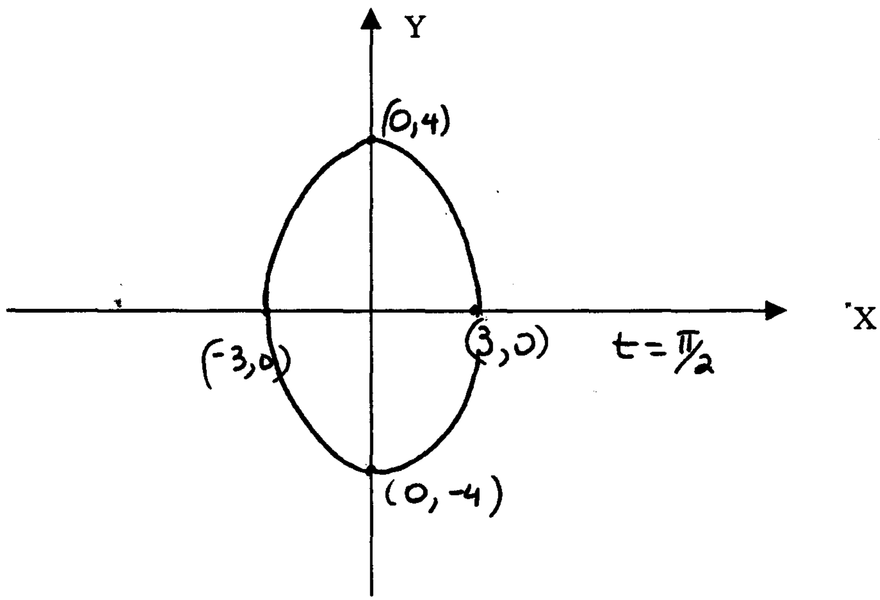
3. An object moves in such a way that its position at any time t is given by $\bar{s}(t) = (2t^2 - 3t + 5)\bar{i} + (3\sin t)\bar{j}$. Find its acceleration at any time t.

$$\vec{v}(t) = s'(t) = (4t - 3)\vec{i} + (3\cos t)\vec{j}$$

$$\vec{a}(t) = v'(t) = s''(t) = 4\vec{i} - 3\sin t\,\vec{j}$$

4. A moving object with a position vector described by $\vec{r}(t) = (3\sin t)\vec{i} + (4\cos t)\vec{j}$.

a. Draw the graph of $\vec{r}(t)$, $-2\pi \le t \le 2\pi$.



b. Confirm that the points (3, 0) and (0, 4) are on the graph.

$x = 3\sin t$

$y = 4\cos t$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

(3,0) and (0,4) satisfy the equation.

c. Write the vector equation for the velocity, $\vec{v}(t)$, and the acceleration vector, $\vec{a}(t)$.

$v(t) = r'(t) = (3\cos t)\vec{i} + (-4\sin t)\vec{j}$

$a(t) = v'(t) = r''(t) = (-3\sin t)\vec{i} + (-4\cos t)\vec{j}$

d. Find the instantaneous velocity vector, speed and acceleration of the object at $t = 0.6$.

$v(0.6) \approx 2.476\vec{i} - 2.259\vec{j}$

$\|v\| = 3.351$ (Rounded values)

$a(0.6) \approx -1.94\vec{i} - 3.301\vec{j}$

AP CALCULUS
UNIT IX – ACTIVITY 2

NAME Key
DATE ____________

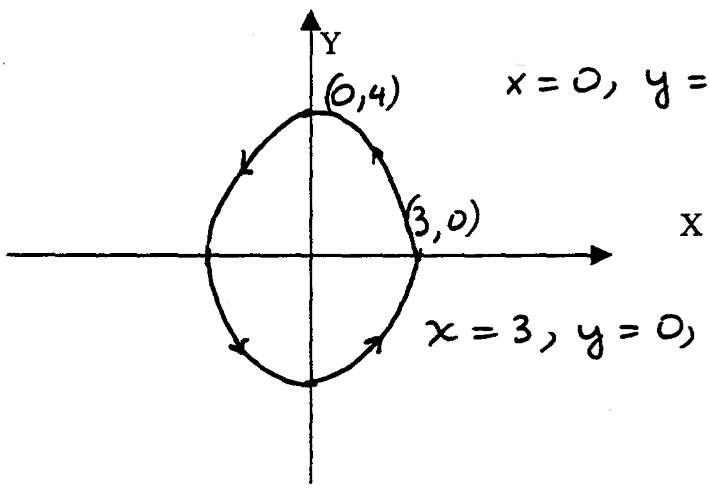
PARAMETRIC AND VECTOR VALUE FUNCTIONS

1. The path of an object B is described by the parametric equations below:

$x = 3\cos 0.6t$

$y = 4\sin 0.6t$

a. Graph the path in the xy-plane, where $0 \le t \le 10.5$. Indicate the direction of the movement. Justify your answer.



$x = 0,\ y = 4,\ t = 5\pi/6$

$x = 3,\ y = 0,\ t = 0$

b. Where is the line tangent to the curve horizontal? Where is the line tangent to the curve vertical? Express your solution in x- and y-coordinates. Justify your answer.

V: (3,0), (−3,0) when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

H: (0,4), (0,−4) when $\frac{dx}{dt} \neq 0$ and $\frac{dy}{dt} = 0$

c. Find the velocity and speed of this object when $t = 2$.

$v(t) = -1.677\vec{i} + .870\vec{j}$ at $t = 2$

Speed: $|v| = 1.889$

d. Find the magnitude of the acceleration at $t = 2$.

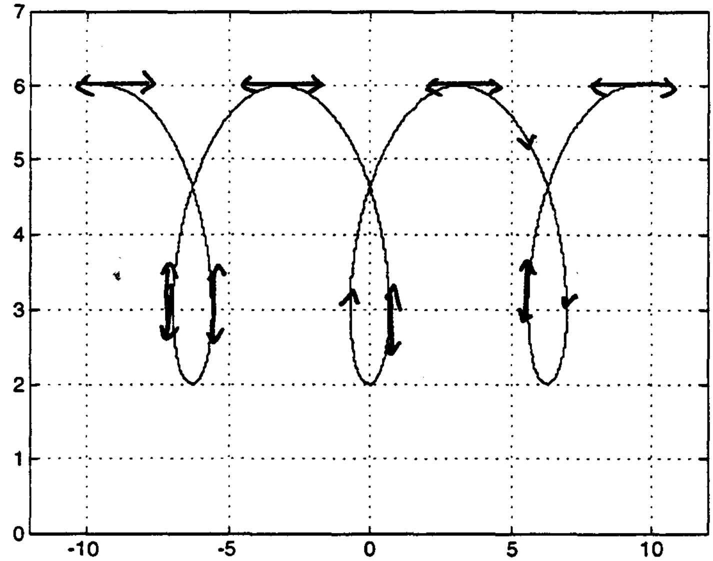
$a(t) = -.391\vec{i} - 1.342\vec{j}$ when $t = 2$

$|a| = 1.398$

2. The path of an object is described by the parametric equations below:

$x = t - 2\sin t$

$y = 4 - 2\cos t$



a. Find the values of t where tangent lines are horizontal or vertical.

H: when $t = k\pi$, $k \in$ Integer

V: when $t = 2k\pi \pm \pi/3$, $k \in$ Integer

b. On the graph, indicate the direction of movement and the location of horizontal and vertical tangent lines.

See graph.

c. Evaluate the speed of this object when $t = 1$.

$|v| = 1.685$

d. Evaluate the magnitude of the acceleration for this object at $t = 1$.

$|a| = 2$

e. When does the object move fastest? Justify your answer.

When speed ($|v|$) is greatest,

$t = \pi + 2k\pi$, $k \in$ Integer

UNIT IX
STUDENT WORKSHEETS

NAME__________________
DATE___________________

PARAMETRIC AND VECTOR VALUE FUNCTIONS

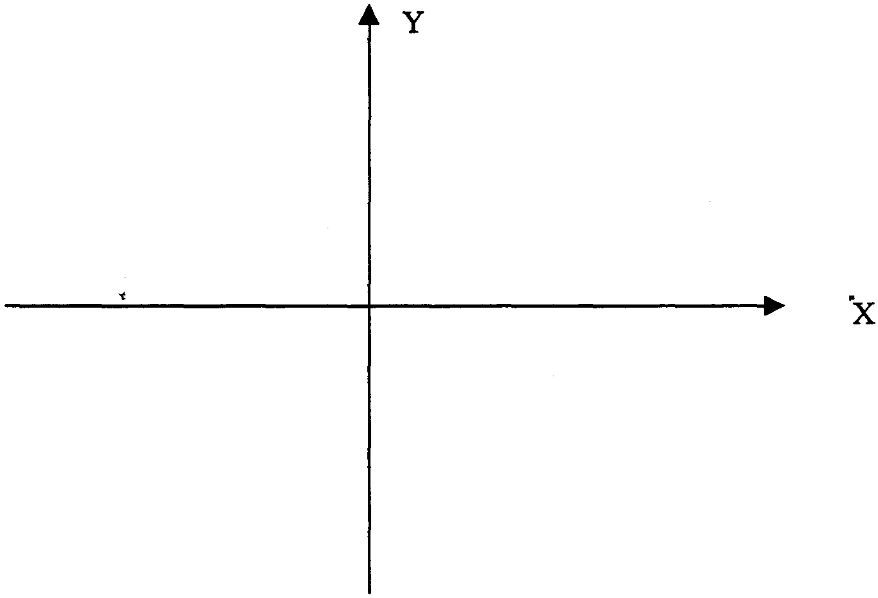
1. If $x = t^2 - 1$ and $y = 2t^3 + 1$, then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

2. If $x = 4\cos 2\theta$ and $y = 5\sin 3\theta$, then find $\frac{d^2y}{dx^2}$.

3. An object moves in such a way that its position at any time t is given by $\bar{s}(t) = (2t^2 - 3t + 5)\bar{i} + (3\sin t)\bar{j}$. Find its acceleration at any time t.

4. A moving object with a position vector described by $\vec{r}(t) = (3\sin t)\vec{i} + (4\cos t)\vec{j}$.

a. Draw the graph of $\vec{r}(t)$, $-2\pi \le t \le 2\pi$.



b. Confirm that the points (3, 0) and (0, 4) are on the graph.

c. Write the vector equation for the velocity, $\vec{v}(t)$, and the acceleration vector, $\vec{a}(t)$.

d. Find the instantaneous velocity vector, speed and acceleration of the object at $t = 0.6$.

NAME________________
DATE ________________

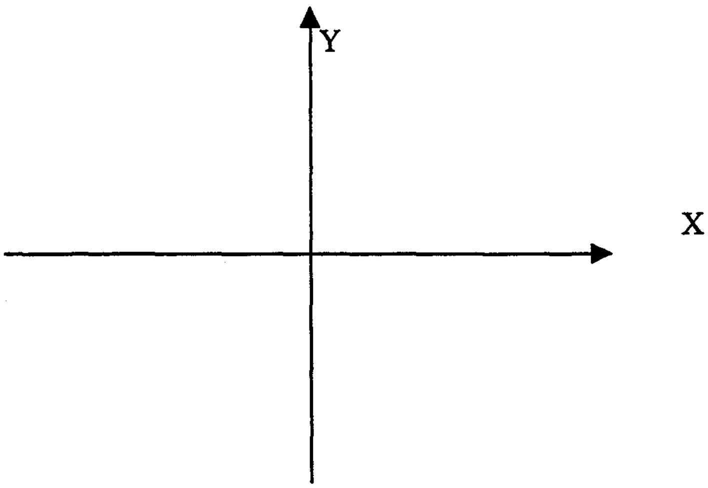
PARAMETRIC AND VECTOR VALUE FUNCTIONS

1. The path of an object B is described by the parametric equations below:

$x = 3\cos 0.6t$

$y = 4\sin 0.6t$

a. Graph the path in the xy-plane, where $0 \le t \le 10.5$. Indicate the direction of the movement. Justify your answer.



b. Where is the line tangent to the curve <u>horizontal</u>? Where is the line tangent to the curve <u>vertical</u>? Express your solution in x- and y-coordinates. Justify your answer.

c. Find the velocity and speed of this object when $t = 2$.

d. Find the magnitude of the acceleration at $t = 2$.

NAME ____________________
DATE ____________________

MULTIPLE CHOICE PRACTICE

1. If f is a vector-valued function defined by $f(t)=\langle \sin t, -e^{-t}\rangle$, then $f''(t)=$

   a) $\sin t - e^{-t}$
   b) $\cos t + e^{-t}$
   c) $\sqrt{\sin^2 t + e^{2t}}$
   d) $\langle -\sin t, -e^{-t}\rangle$
   e) $\langle \sin t, e^{-t}\rangle$

2. The area inside $r=4\sin\theta$ and outside $r=2\sin\theta$ is given by

   a) $\int_0^{\pi} \sin^2\theta \, d\theta$
   b) $4\int_0^{\pi} \sin^2\theta \, d\theta$
   c) $6\int_0^{\pi} \sin^2\theta \, d\theta$
   d) $\int_0^{2\pi} \sin^2\theta \, d\theta$
   e) $6\int_0^{2\pi} \sin^2\theta \, d\theta$

3. If an object's acceleration vector is $\vec{a}(t)=4e^{2t}\,\vec{i}+2\,\vec{j}$, $v(0)=0$, and $s(0)=0$, what is its position vector?

   a) $2e^{2t}\,\vec{i}+2t\,\vec{j}$
   b) $(e^{2t}-2t-1)\,\vec{i}+t^2\,\vec{j}$
   c) $(e^{2t}-1)\,\vec{i}+t^2\,\vec{j}$
   d) $4e^{2t}+2t$
   e) $2e^{2t}+2t$

4. The speed at $t=0$ of the vector in #3 is

   a) 0
   b) 1
   c) 2
   d) $2e^2$
   e) $2e^2+2$

5. The area inside one loop of $r=4\sin 2\theta$ is given by

   a) $16\int_0^{2\pi} \sin 2\theta \, d\theta$
   b) $8\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$
   c) $16\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$
   d) $8\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$
   e) $8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$

AP CALCULUS
UNIT IX

NAME Key.
DATE

MULTIPLE CHOICE PRACTICE

1. If f is a vector-valued function defined by $f(t) = \langle \sin t, -e^{-t} \rangle$, then $f''(t) =$

   a) $\sin t - e^{-t}$
   b) $\cos t + e^{-t}$
   c) $\sqrt{\sin^2 t + e^{2t}}$
   d) $\langle -\sin t, -e^{-t} \rangle$
   e) $\langle \sin t, e^{-t} \rangle$

   (D)

2. The area inside $r = 4\sin\theta$ and outside $r = 2\sin\theta$ is given by

   a) $\int_0^{\pi} \sin^2\theta \, d\theta$
   b) $4\int_0^{\pi} \sin^2\theta \, d\theta$
   c) $6\int_0^{\pi} \sin^2\theta \, d\theta$
   d) $\int_0^{2\pi} \sin^2\theta \, d\theta$
   e) $6\int_0^{2\pi} \sin^2\theta \, d\theta$

   (C)

3. If an object's acceleration vector is $\vec{a}(t) = 4e^{2t}\,\vec{\mathbf{i}} + 2\,\vec{\mathbf{j}}$, $v(0) = 0$, and $s(0) = 0$, what is its position vector?

   a) $2e^{2t}\,\vec{\mathbf{i}} + 2t\,\vec{\mathbf{j}}$
   b) $(e^{2t} - 2t - 1)\,\vec{\mathbf{i}} + t^2\,\vec{\mathbf{j}}$
   c) $(e^{2t} - 1)\,\vec{\mathbf{i}} + t^2\,\vec{\mathbf{j}}$
   d) $4e^{2t} + 2t$
   e) $2e^{2t} + 2t$

   (B)

4. The speed at $t = 0$ of the vector in #3 is

   a) 0
   b) 1
   c) 2
   d) $2e^2$
   e) $2e^2 + 2$

   (A)

5. The area inside one loop of $r = 4\sin 2\theta$ is given by

   a) $16\int_0^{2\pi} \sin 2\theta \, d\theta$
   b) $8\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$
   c) $16\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$
   d) $8\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$
   e) $8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$

   (B)

1. The area of a sector of a circle with radius r and central angle θ is __?__.

2. Solve for y:

$$\frac{dy}{dx} = x^3 + 3^x$$

3. If $\vec{r} = \sin^2 t\,\vec{i} + \sec t\,\vec{j}$ is a position vector, find $\vec{v}$, the velocity vector.

4. $x = \tan^{-1} t \qquad y = \ln t$

$$\frac{dy}{dx}\Big|_{t=2} =$$

5. Write an expression for the Length of the curve from X=1 to x=3 if

f(x) = tan x

1. The area of a sector of a circle with radius and central angle θ is __?__.

2. Solve for y:

$$\frac{dy}{dx} = x^3 + 3^x$$

3. If $\vec{r} = \sin^2 t\,\vec{i} + \sec t\,\vec{j}$ is a position vector, find $\vec{v}$, the velocity vector.

4. $x = \tan^{-1} t \qquad y = \ln t$

$$\left.\frac{dy}{dx}\right|_{t=2} =$$

5. Write an expression for the length of the curve from x=1 to x=3 if f(x)=tan x

---

1. $A = \frac{1}{2} r^2 \theta$

2.

$$dy = (x^3 + 3^x)dx$$

$$y = \frac{x^4}{4} + \frac{3^x}{\ln 3} + C$$

3.

$$\vec{v} = 2\sin t \cos t\,\vec{i} + \sec t \tan t\,\vec{j}$$

$$\vec{v} = \sin 2t\,\vec{i} + \sec t \tan t\,\vec{j}$$

4. 2.5

$$\int_1^3 \sqrt{1 + \sec^4 x}\,dx$$

1. If a velocity vector is $\langle 2\sin t, \cos t \rangle$, then a particle's speed at $t = \dfrac{\pi}{6}$ is ______.

in folder

2. For the curve defined by $x = 2\cos t \; and \; y = 4\sin t$ find slope of tangent to the curve at $t = \dfrac{\pi}{3}$

3. Rewrite as an expression with an integer base.
$15e^{\ln\frac{2}{5}t}$

4. Find $\dfrac{dy}{dx}$ when
$x + y = \sin(xy)$

5. Graph: $r = 2 + \cos\theta$
On polar coordinates

1. If a velocity vector is $\langle 2\sin t, \cos t\rangle$, then a particle's speed at $t = \dfrac{\pi}{6}$ is ____.

2. For the curve defined by $x = 2\cos t \ and \ y = 4\sin t$ find slope of tangent to the curve at $t = \dfrac{\pi}{3}$

3. Rewrite as an expression with an integer base.
$15e^{\ln\frac{2}{5}t}$

4. Find $\dfrac{dy}{dx}$ when $x + y = \sin(xy)$

5. Graph: $r = 1 + \cos\theta$
On polar coordinates

1. $\sqrt{4\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{6}}$

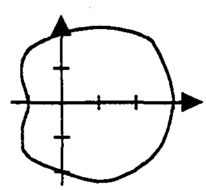
$\sqrt{2.5}$

2. $\dfrac{dy}{dx} = \dfrac{4\cos t}{-2\sin t}$

$\dfrac{-2}{\sqrt{3}}$

3. $15(2)^{\frac{t}{5}}$

4. $\dfrac{dy}{dx} = \dfrac{y\cos(xy) - 1}{1 - x\cos(xy)}$

5. 

1. Graph $r = \cos^2\theta$ on polar coordinates

2. $\dfrac{d}{dx}\displaystyle\int_3^{\arccos x} e^t\,dt$

3. An integral for the area inside the polar curve $r = 6\sin\theta$ and outside the polar curve $r = 3$ is

4. Graph the slope field for $\dfrac{dy}{dx} = \dfrac{x}{y^2}$ on the lattice points for [-2,2]x[1,2]

5. If $v(t) = \cos^3 3t$, write an integral expression for the total distance from $t = 1$ to $t = 12$.

1. Graph $r = 9\cos^2\theta$ on polar coordinates

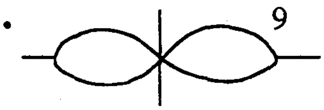
2. $\dfrac{d}{dx}\displaystyle\int_3^{\arccos x} e^t\,dt$

3. An integral for the area inside the polar curve $r = 6\sin\theta$ and outside the polar curve $r = 3$ is

4. Graph the slope field for $\dfrac{dy}{dx} = \dfrac{x}{y^2}$ on the lattice points for [-2,2]x[1,2]

5. If $v(t) = \cos^3 3t$, write an integral expression for the total distance from $t = 1$ to $t = 12$.

1.



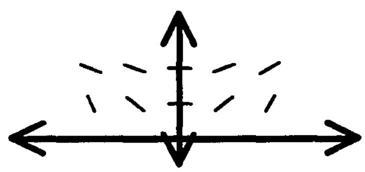
2.

$$-e^{\arccos x}\,\frac{1}{\sqrt{1-x^2}}$$

3.

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}[(6\sin\theta)^2 - 3^2]\,d\theta$$

4.



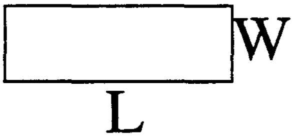
5.

$$\int_1^{12} \left|\cos^3 3t\right|\,dt$$

1. $g(x) = \int h(x)dx$
   if and only if_____.

2. Graph: $r = 2\sin 3\theta$
   (polar coordinates)

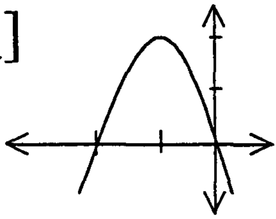
3. Set up an integral for the area inside one loop in the graph of #2.

4. Given: 

$$\frac{dL}{dt} = 8\frac{cm}{\sec} \; ; \; \frac{dW}{dt} = -5\frac{cm}{\sec}$$

How fast is the area changing when L=40cm and W=20cm?

5. Sketch the graph of the derivative of f(x) [graphed ⇓]

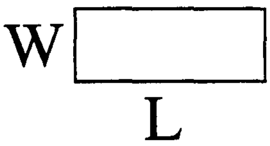


1. $g(x) = \int h(x)dx$
if and only if_____.

2. The acceleration vector for a particle is $\vec{a}(t) = 6t\,\vec{i} + 3t^2\vec{j}$. Its velocity is 0 when t=0, Find the velocity vector.

3. Set up an integral for the area inside one loop in the graph of $r = 2\sin 3\theta$.

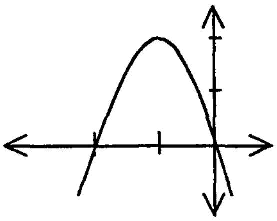
4. Given:



$$\frac{dL}{dt} = 8\frac{cm}{\sec}\ ;\ \frac{dW}{dt} = -5\frac{cm}{\sec}$$

How fast is the area changing when L=40cm and W=20cm?

5. Sketch the graph of the derivative of f(x) [graphed ⇓]

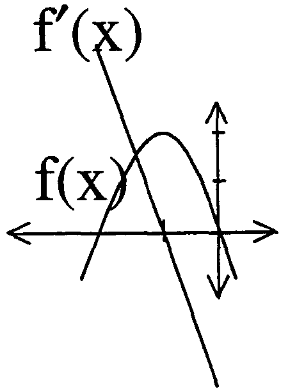


1. $g'(x) = h(x$

2.
$v(t) = 3t^2\vec{i} + t^3$

$$\int_0^{\frac{\pi}{3}} \frac{1}{2}(2\sin 3\theta)^2 d$$

4
$-40\,{}^{cm}\!/\!_{\min^2}$



1. Set up an integral for the volume generated whenf(x)=sinx is revolved about the x-axis from x=1 to x=2

2. If $f(x)=\frac{x^2}{2}+\frac{2x^3}{3}+\frac{3x^4}{4}+\ldots$ then $f'(x)=$

3. Complete the table for Euler's method if $\frac{dy}{dx}=\frac{x}{y^2}$

x	y	dy/dx	dy
1	2	a)____	b)____
1.1	c) ____		

4. $\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}$

5. $f(x)=\sin^{-1}e^x$

1. Set up an integral for the volume generated whenf(x)=sinx is revolved about the x-axis from x=1 to x=2

2. If $f(x)=\frac{x^2}{2}+\frac{2x^3}{3}+\frac{3x^4}{4}+...$ then $f'(x)=$

3. Complete the table for Euler's method if $\frac{dy}{dx}=\frac{x}{y^2}$

x	y	dy/dx	dy
1	2	a)____	b)____
1.1	c) ____		

4. $\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}$

5. $f(x)=\sin^{-1}e^x$
$f'(x)=$

ANSWERS:

1. $\int_1^2 \pi\sin^2 x dx$

2. $x+2x^2+3x^3+..$

3. a) 0.25
b) 0.025
c) 2.025

4. $\lim_{h\to 0}\frac{\sqrt{9+h}-\sqrt{9}}{h}$
$f(x)=\sqrt{x}$
$f'(x)=\frac{1}{2\sqrt{x}}$
$f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$

5. $\frac{e^x}{\sqrt{1+e^{2x}}}$